Cheat Sheet J - Verification

Pensieve header: Cheat Sheet J Verification; continues pensieve://2013-04/; continued pensieve://2014-01/.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2013-12"];
<< "../2013-05/FreeLie.m";
tm[u_, v_, w_] := LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩];
CC[u_, γ_LieSeries] := LieMorphism[u → Ad[γ][u]];
CC_u_[γ_] := CC[u, γ];
RC[u_, γ_LieSeries, ub_][ser_] :=
  StableApply[LieMorphism[⟨u⟩ → Ad[γ][⟨ub⟩]], ser];
RC[u_, γ_LieSeries][ser_] := ser // RC[u, γ, ⟨v⟩] // LieMorphism[⟨v⟩ → ⟨u⟩];
RC_u_[γ_] := RC[u, γ];
Print /@ {{t = ⟨"t"⟩, u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩},
    α = RandomLieSeries[{t, u, v}],
    β = RandomLieSeries[{t, u, v}],
    γ = RandomLieSeries[{t, u, v}]
  };
$SeriesShowDegree = 3; $SeriesCompareDegree = 6;
```

```
{⟨t⟩, ⟨u⟩, ⟨v⟩, ⟨w⟩}
```

$$\text{LS}\left[-2\,\overline{u}+2\,\overline{v},\ \overline{tu}-\frac{3\,\overline{tv}}{2}+\frac{3\,\overline{uv}}{2},\ \frac{5}{6}\,\overline{t\,\overline{tu}}+\frac{1}{3}\,\overline{t\,\overline{tv}}+\frac{5}{6}\,\overline{t\,\overline{uv}}+\overline{u\,\overline{uv}}+\overline{\overline{tu}\,u}+\frac{11}{6}\,\overline{\overline{tv}\,u}+\frac{4}{3}\,\overline{\overline{tv}\,v}\right]$$

$$\text{LS}\left[-\overline{t}+\overline{u}-2\,\overline{v},\ -\frac{\overline{tu}}{2}+\overline{tv},\ \frac{11}{6}\,\overline{t\,\overline{tu}}+2\,\overline{t\,\overline{tv}}-\frac{1}{6}\,\overline{t\,\overline{uv}}-\frac{1}{3}\,\overline{u\,\overline{uv}}+\frac{4}{3}\,\overline{\overline{tu}\,u}-\frac{1}{6}\,\overline{\overline{tv}\,u}-\frac{1}{6}\,\overline{\overline{tv}\,v}+\frac{11}{6}\,\overline{\overline{uv}\,v}\right]$$

$$\text{LS}\left[-2\,\overline{t},\ -\frac{\overline{tu}}{2}+\frac{3\,\overline{tv}}{2}+\frac{3\,\overline{uv}}{2},\ 2\,\overline{t\,\overline{tv}}+\frac{2}{3}\,\overline{t\,\overline{uv}}-\frac{5}{6}\,\overline{u\,\overline{uv}}-\overline{\overline{tu}\,u}-\frac{1}{3}\,\overline{\overline{tv}\,u}-\frac{5}{3}\,\overline{\overline{tv}\,v}+\frac{4}{3}\,\overline{\overline{uv}\,v}\right]$$

```
ad[u_, γ_LieSeries] := LieDerivation[u → b[γ, u]];
ad_u_[γ_] := ad[u, γ];
ϵ /: ϵ^2 = 0;
```

- Some preliminary testing

```
(α // RC_u[γ] // CC_u[-γ]) ≡ α
```

```
True
```

```
(α // RC_u[γ] // RC_u[-γ // RC_u[γ]]) ≡ α
```

```
True
```

1. The Definition of J

```
J[u_, γ_] := Module[{s}, ∫_0^1 (γ // RC_u[s γ] // div_u // CC_u[-s γ]) ds];
J_u_[γ_] := J[u, γ];
```

```
Ju[α][{4}]
```

$$\mathrm{CWS}\left[\widehat{u},\ -\widehat{tu}+\frac{\widehat{uv}}{2},\ -\frac{5\,\widehat{ttu}}{3}-\frac{\widehat{tuu}}{3}+\frac{3\,\widehat{tuv}}{4}-\frac{3\,\widehat{tvu}}{4}+\frac{\widehat{uuv}}{4}-\frac{5\,\widehat{uvv}}{3},\right.$$
$$-\frac{25\,\widehat{tttu}}{24}-\frac{13\,\widehat{ttuu}}{24}-\frac{13\,\widehat{ttuv}}{8}+\frac{7\,\widehat{ttvu}}{3}+\frac{\widehat{tutu}}{12}-\frac{2\,\widehat{tutv}}{3}+\frac{\widehat{tuuu}}{24}+\frac{17\,\widehat{tuuv}}{12}-\frac{\widehat{tuvu}}{6}+$$
$$\left.\frac{3\,\widehat{tuvv}}{8}-\frac{5\,\widehat{tvuu}}{6}+\frac{37\,\widehat{tvuv}}{24}+\frac{\widehat{tvvu}}{3}+\frac{4\,\widehat{uuuv}}{3}-\frac{7\,\widehat{uuvv}}{4}+\frac{13\,\widehat{uvuv}}{4}-\frac{13\,\widehat{uvvv}}{8}\right]$$

2. The J_{uv} equation

```
Print /@ {
    0 → {α, β},
    1 → (t1 = Ju[α] + (Jv[β // RCu[α]] // CCu[-α])),
    2 → (t2 = Jv[β] + (Ju[α // RCv[β]] // CCv[-β])),
    3 →  t1 ≡ t2
   };
```

$$0 \to \left\{\mathrm{LS}\left[\overline{u}+2\,\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}+\frac{\overline{uv}}{2},\right.\right.$$
$$\left.-\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{5}{3}\,\overline{t\,\overline{tv}}-\overline{t\,\overline{uv}}-\frac{1}{3}\,\overline{u\,\overline{uv}}+\frac{5}{6}\,\overline{tu\,u}-2\,\overline{tv\,u}+\frac{1}{2}\,\overline{tv\,v}-\frac{11}{6}\,\overline{uv\,v}\right],$$
$$\left.\mathrm{LS}\left[-\overline{u},\ -2\,\overline{tv}+\frac{\overline{uv}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{tu}}-\frac{1}{6}\,\overline{t\,\overline{tv}}-\frac{1}{3}\,\overline{t\,\overline{uv}}-\frac{5}{6}\,\overline{u\,\overline{uv}}+2\,\overline{tu\,u}-2\,\overline{tv\,v}+\frac{11}{6}\,\overline{uv\,v}\right]\right\}$$

$$1 \to \mathrm{CWS}\left[\widehat{u},\ -\widehat{tu}-2\,\widehat{tv}+3\,\widehat{uv},\ -\frac{5\,\widehat{ttu}}{3}-\frac{\widehat{ttv}}{6}-\frac{\widehat{tuu}}{3}+\frac{5\,\widehat{tuv}}{12}+\frac{3\,\widehat{tvu}}{4}+2\,\widehat{tvv}-\frac{7\,\widehat{uuv}}{3}-\frac{\widehat{uvv}}{2}\right]$$

$$2 \to \mathrm{CWS}\left[\widehat{u},\ -\widehat{tu}-2\,\widehat{tv}+3\,\widehat{uv},\ -\frac{5\,\widehat{ttu}}{3}-\frac{\widehat{ttv}}{6}-\frac{\widehat{tuu}}{3}+\frac{5\,\widehat{tuv}}{12}+\frac{3\,\widehat{tvu}}{4}+2\,\widehat{tvv}-\frac{7\,\widehat{uuv}}{3}-\frac{\widehat{uvv}}{2}\right]$$

$$3 \to \mathrm{True}$$

```
Print /@ {
    0 → {α, β},
    1 → (t1 = Ju[α] + (Jv[β // RCu[s α]] // CCu[-s α])),
    2 → (t2 = Jv[β] + (Ju[α // RCv[s β]] // CCv[-s β])),
    3 →  t1 ≡ t2
   };
```

$$0 \to \left\{ LS\left[\overline{u} + 2\,\overline{v},\ -\overline{tu} + \frac{\overline{tv}}{2} + \frac{\overline{uv}}{2},\right.\right.$$
$$\left. -\frac{5}{3}\,\overline{t\,\overline{tu}} + \frac{5}{3}\,\overline{t\,\overline{tv}} - \overline{t\,\overline{uv}} - \frac{1}{3}\,\overline{u\,\overline{uv}} + \frac{5}{6}\,\overline{\overline{tu}\,u} - 2\,\overline{\overline{tv}\,u} + \frac{1}{2}\,\overline{\overline{tv}\,v} - \frac{11}{6}\,\overline{\overline{uv}\,v}\right],$$
$$\left. LS\left[-\overline{u},\ -2\,\overline{tv} + \frac{\overline{uv}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{tu}} - \frac{1}{6}\,\overline{t\,\overline{tv}} - \frac{1}{3}\,\overline{t\,\overline{uv}} - \frac{5}{6}\,\overline{u\,\overline{uv}} + 2\,\overline{\overline{tu}\,u} - 2\,\overline{\overline{tv}\,v} + \frac{11}{6}\,\overline{\overline{uv}\,v}\right]\right\}$$

$$1 \to CWS\left[\widehat{u},\ -\widehat{tu} - 2\,\widehat{tv} + \widehat{uv} + 2\,s\,\widehat{uv},\ -\frac{5\,\widehat{ttu}}{3} - \frac{\widehat{ttv}}{6} - \frac{\widehat{tuu}}{3} + \frac{5\,\widehat{tuv}}{12} +\right.$$
$$\left. \frac{\widehat{tvu}}{4} + \frac{s\,\widehat{tvu}}{2} + 2\,\widehat{tvv} - \frac{5\,\widehat{uuv}}{6} - \frac{s\,\widehat{uuv}}{2} - s^2\,\widehat{uuv} - \frac{7\,\widehat{uvv}}{2} + s\,\widehat{uvv} + 2\,s^2\,\widehat{uvv}\right]$$

$$2 \to CWS\left[\widehat{u},\ -\widehat{tu} - 2\,\widehat{tv} + \widehat{uv} + 2\,s\,\widehat{uv},\right.$$
$$\left. -\frac{5\,\widehat{ttu}}{3} - \frac{\widehat{ttv}}{6} - \frac{\widehat{tuu}}{3} + \frac{5\,\widehat{tuv}}{12} + \frac{\widehat{tvu}}{4} + \frac{s\,\widehat{tvu}}{2} + 2\,\widehat{tvv} - \frac{5\,\widehat{uuv}}{6} - \frac{s\,\widehat{uuv}}{2} - s^2\,\widehat{uuv} - \frac{7\,\widehat{uvv}}{2} + 3\,s\,\widehat{uvv}\right]$$

$$3 \to -\frac{5\,CW[ttu]}{3} - \frac{CW[ttv]}{6} - \frac{CW[tuu]}{3} + \frac{5\,CW[tuv]}{12} + \frac{CW[tvu]}{4} + \frac{1}{2}\,s\,CW[tvu] + 2\,CW[tvv] -$$
$$\frac{5\,CW[uuv]}{6} - \frac{1}{2}\,s\,CW[uuv] - s^2\,CW[uuv] - \frac{7\,CW[uvv]}{2} + s\,CW[uvv] + 2\,s^2\,CW[uvv] ==$$
$$-\frac{5\,CW[ttu]}{3} - \frac{CW[ttv]}{6} - \frac{CW[tuu]}{3} + \frac{5\,CW[tuv]}{12} + \frac{CW[tvu]}{4} + \frac{1}{2}\,s\,CW[tvu] +$$
$$2\,CW[tvv] - \frac{5\,CW[uuv]}{6} - \frac{1}{2}\,s\,CW[uuv] - s^2\,CW[uuv] - \frac{7\,CW[uvv]}{2} + 3\,s\,CW[uvv]$$

```
Print /@ {
   0 → {α, β},
   1 → (t1 = Ju[α]),
   2 → (t2 = Ju[α // RCv[β]] // CCv[-β]),
   3 → t1 ≡ t2
  };
```

$$0 \to \left\{ LS\left[\overline{u} + 2\,\overline{v},\ -\overline{tu} + \frac{\overline{tv}}{2} + \frac{\overline{uv}}{2},\right.\right.$$
$$\left. -\frac{5}{3}\,\overline{t\,\overline{tu}} + \frac{5}{3}\,\overline{t\,\overline{tv}} - \overline{t\,\overline{uv}} - \frac{1}{3}\,\overline{u\,\overline{uv}} + \frac{5}{6}\,\overline{\overline{tu}\,u} - 2\,\overline{\overline{tv}\,u} + \frac{1}{2}\,\overline{\overline{tv}\,v} - \frac{11}{6}\,\overline{\overline{uv}\,v}\right],$$
$$\left. LS\left[-\overline{u},\ -2\,\overline{tv} + \frac{\overline{uv}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{tu}} - \frac{1}{6}\,\overline{t\,\overline{tv}} - \frac{1}{3}\,\overline{t\,\overline{uv}} - \frac{5}{6}\,\overline{u\,\overline{uv}} + 2\,\overline{\overline{tu}\,u} - 2\,\overline{\overline{tv}\,v} + \frac{11}{6}\,\overline{\overline{uv}\,v}\right]\right\}$$

$$1 \to CWS\left[\widehat{u},\ -\widehat{tu} + \frac{\widehat{uv}}{2},\ -\frac{5\,\widehat{ttu}}{3} - \frac{\widehat{tuu}}{3} + \frac{3\,\widehat{tuv}}{4} - \frac{3\,\widehat{tvu}}{4} + \frac{\widehat{uuv}}{4} - \frac{5\,\widehat{uvv}}{3}\right]$$

$$2 \to CWS\left[\widehat{u},\ -\widehat{tu} + \frac{5\,\widehat{uv}}{2},\ -\frac{5\,\widehat{ttu}}{3} - \frac{\widehat{tuu}}{3} + \frac{3\,\widehat{tuv}}{4} - \frac{\widehat{tvu}}{4} - \frac{5\,\widehat{uuv}}{4} + \frac{4\,\widehat{uvv}}{3}\right]$$

$$3 \to -CW[tu] + \frac{CW[uv]}{2} == -CW[tu] + \frac{5\,CW[uv]}{2}$$

```
Print /@ {
    0 → {α, β},
    1 → (t1 = Ju[α] // RCu[α] // RCv[β] // RCu[α]),
    2 → (t2 = Ju[α // RCv[β]] // RCu[α // RCv[β]]),
    3 → t1 ≡ t2
  };
```

$$0 \to \left\{\mathrm{LS}\left[\overline{u}+2\,\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}+\frac{\overline{uv}}{2},\right.\right.$$
$$\left.-\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{5}{3}\,\overline{t\,\overline{tv}}-\overline{t\,\overline{uv}}-\frac{1}{3}\,\overline{u\,\overline{uv}}+\frac{5}{6}\,\overline{\overline{tu}\,u}-2\,\overline{\overline{tv}\,u}+\frac{1}{2}\,\overline{\overline{tv}\,v}-\frac{11}{6}\,\overline{\overline{uv}\,v}\right],$$
$$\left.\mathrm{LS}\left[-\overline{u},\ -2\,\overline{tv}+\frac{\overline{uv}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{tu}}-\frac{1}{6}\,\overline{t\,\overline{tv}}-\frac{1}{3}\,\overline{t\,\overline{uv}}-\frac{5}{6}\,\overline{u\,\overline{uv}}+2\,\overline{\overline{tu}\,u}-2\,\overline{\overline{tv}\,v}+\frac{11}{6}\,\overline{\overline{uv}\,v}\right]\right\}$$

$$1 \to \mathrm{CWS}\left[\widehat{u},\ -\widehat{tu}+\frac{\widehat{uv}}{2},\ -\frac{5\,\widehat{ttu}}{3}-\frac{\widehat{tuu}}{3}+\frac{19\,\widehat{tuv}}{4}-\frac{19\,\widehat{tvu}}{4}+\frac{\widehat{uuv}}{4}-\frac{5\,\widehat{uvv}}{3}\right]$$

$$2 \to \mathrm{CWS}\left[\widehat{u},\ -\widehat{tu}+\frac{5\,\widehat{uv}}{2},\ -\frac{5\,\widehat{ttu}}{3}-\frac{\widehat{tuu}}{3}+\frac{11\,\widehat{tuv}}{4}-\frac{9\,\widehat{tvu}}{4}-\frac{5\,\widehat{uuv}}{4}+\frac{4\,\widehat{uvv}}{3}\right]$$

$$3 \to -\mathrm{CW}[tu]+\frac{\mathrm{CW}[uv]}{2} == -\mathrm{CW}[tu]+\frac{5\,\mathrm{CW}[uv]}{2}$$

3. The t equation

```
Print /@ {
    0 → γ,
    1 → (t1 = J[w, γ // tm[u, v, w]]),
    2 → (t2 = Ju[γ] // tm[u, v, w]),
    3 → (t3 = Jv[γ // RCu[γ]] // CCu[-γ] // tm[u, v, w]),
    4 → t1 ≡ t2 + t3
  };
```

$$0 \to \mathrm{LS}\left[0,\ -2\,\overline{tu}+2\,\overline{tv}+2\,\overline{uv},\ -\frac{2}{3}\,\overline{t\,\overline{tu}}-\frac{3}{2}\,\overline{t\,\overline{tv}}-\frac{1}{6}\,\overline{t\,\overline{uv}}-2\,\overline{\overline{tu}\,u}+\frac{1}{6}\,\overline{\overline{tv}\,u}-2\,\overline{\overline{tv}\,v}-\frac{5}{6}\,\overline{\overline{uv}\,v}\right]$$

$$1 \to \mathrm{CWS}\left[0,\ 0,\ -\frac{13\,\widehat{ttw}}{6}+\frac{23\,\widehat{tww}}{6}\right]$$

$$2 \to \mathrm{CWS}\left[0,\ -2\,\widehat{tw}-2\,\widehat{ww},\ -\frac{2\,\widehat{ttw}}{3}+\frac{13\,\widehat{tww}}{6}-\frac{5\,\widehat{www}}{6}\right]$$

$$3 \to \mathrm{CWS}\left[0,\ 2\,\widehat{tw}+2\,\widehat{ww},\ -\frac{3\,\widehat{ttw}}{2}+\frac{5\,\widehat{tww}}{3}+\frac{5\,\widehat{www}}{6}\right]$$

$$4 \to \mathrm{True}$$

```
Print /@ {
    0 → {γ, γw = γ // tm[u, v, w]},
    1 → (t1 = J[w, γw] // RCw[γw]),
    2 → (t2 = Ju[γ] // tm[u, v, w] // RCw[γw]),
    3 → (t3 = Jv[γ // RCu[γ]] // RCv[γ // RCu[γ]] // tm[u, v, w]),
    4 → t1 ≡ t2 + t3
  };
```

$$0 \to \left\{ \mathrm{LS}\left[0, -2\,\overline{tu} + 2\,\overline{tv} + 2\,\overline{uv}, -\frac{2}{3}\,\overline{ttu} - \frac{3}{2}\,\overline{ttv} - \frac{1}{6}\,\overline{tuv} - 2\,\overline{tuu} + \frac{1}{6}\,\overline{tvu} - 2\,\overline{tvv} - \frac{5}{6}\,\overline{uvv}\right], \right.$$
$$\left. \mathrm{LS}\left[0, 0, -\frac{13}{6}\,\overline{ttw} - \frac{23}{6}\,\overline{tww}\right] \right\}$$

$$1 \to \mathrm{CWS}\left[0, 0, -\frac{13\,\widetilde{ttw}}{6} + \frac{23\,\widetilde{tww}}{6}\right]$$

$$2 \to \mathrm{CWS}\left[0, -2\,\widetilde{tw} - 2\,\widetilde{ww}, -\frac{2\,\widetilde{ttw}}{3} + \frac{13\,\widetilde{tww}}{6} - \frac{5\,\widetilde{www}}{6}\right]$$

$$3 \to \mathrm{CWS}\left[0, 2\,\widetilde{tw} + 2\,\widetilde{ww}, -\frac{3\,\widetilde{ttw}}{2} + \frac{5\,\widetilde{tww}}{3} + \frac{5\,\widetilde{www}}{6}\right]$$

$$4 \to \mathrm{True}$$

- Splitting the t equation

```
Print /@ {
   0 → {γ, γw = γ // tm[u, v, w]},
   1 → (t1 = γw // RC_w[s γw]),
   2 → (t2 = γ // RC_u[s γ] // RC_v[s γ // RC_u[s γ]] // tm[u, v, w]),
   3 → t1 ≡ t2,
   4 → (t3 = γ // RC_u[s γ] // RC_v[s γ // RC_u[s γ]] // tm[u, v, w] // div_w),
   5 → (t4 = γ // RC_u[s γ] // RC_v[s γ // RC_u[s γ]] // div_u // tm[u, v, w]),
   6 → (t5 = γ // RC_u[s γ] // RC_v[s γ // RC_u[s γ]] // div_v // tm[u, v, w]),
   7 → t3 ≡ t4 + t5,
   8 → (t6 = ∫_0^1 (t3 // CC_w[-s γw]) ds),
   9 → t6 ≡ J_w[γw],
   10 → (t7 = ∫_0^1 (t4 // CC_w[-s γw]) ds),
   11 → (t8 = ∫_0^1 (t5 // CC_w[-s γw]) ds),
   12 → t6 ≡ t7 + t8,
   (J_u[γ] // tm[u, v, w] // RC_w[γw]) ≡ (t7 // RC_w[γw])
  };
```

$$0 \to \left\{ LS\left[0, -2\,\overline{tu} + 2\,\overline{tv} + 2\,\overline{uv}, -\frac{2}{3}\,\overline{t\,\overline{tu}} - \frac{3}{2}\,\overline{t\,\overline{tv}} - \frac{1}{6}\,\overline{t\,\overline{uv}} - 2\,\overline{\overline{tu}\,u} + \frac{1}{6}\,\overline{\overline{tv}\,u} - 2\,\overline{\overline{tv}\,v} - \frac{5}{6}\,\overline{\overline{uv}\,v}\right], \right.$$
$$\left. LS\left[0, 0, -\frac{13}{6}\,\overline{t\,\overline{tw}} - \frac{23}{6}\,\overline{\overline{tw}\,w}\right]\right\}$$

$$1 \to LS\left[0, 0, -\frac{13}{6}\,\overline{t\,\overline{tw}} - \frac{23}{6}\,\overline{\overline{tw}\,w}\right]$$

$$2 \to LS\left[0, 0, -\frac{13}{6}\,\overline{t\,\overline{tw}} - \frac{23}{6}\,\overline{\overline{tw}\,w}\right]$$

$$3 \to \text{True}$$

$$4 \to CWS\left[0, 0, -\frac{13\,\widehat{ttw}}{6} + \frac{23\,\widehat{tww}}{6}\right]$$

$$5 \to CWS\left[0, -2\,\widehat{tw} - 2\,\widehat{ww}, -\frac{2\,\widehat{ttw}}{3} + \frac{13\,\widehat{tww}}{6} - \frac{5\,\widehat{www}}{6}\right]$$

$$6 \to CWS\left[0, 2\,\widehat{tw} + 2\,\widehat{ww}, -\frac{3\,\widehat{ttw}}{2} + \frac{5\,\widehat{tww}}{3} + \frac{5\,\widehat{www}}{6}\right]$$

$$7 \to \text{True}$$

$$8 \to CWS\left[0, 0, -\frac{13\,\widehat{ttw}}{6} + \frac{23\,\widehat{tww}}{6}\right]$$

$$9 \to \text{True}$$

$$10 \to CWS\left[0, -2\,\widehat{tw} - 2\,\widehat{ww}, -\frac{2\,\widehat{ttw}}{3} + \frac{13\,\widehat{tww}}{6} - \frac{5\,\widehat{www}}{6}\right]$$

$$11 \to CWS\left[0, 2\,\widehat{tw} + 2\,\widehat{ww}, -\frac{3\,\widehat{ttw}}{2} + \frac{5\,\widehat{tww}}{3} + \frac{5\,\widehat{www}}{6}\right]$$

$$12 \to \text{True}$$

$$-\frac{7\,CW[tttw]}{12} - \frac{17\,CW[ttww]}{24} - \frac{4\,CW[twtw]}{3} - 8\,CW[twww] - \frac{13\,CW[wwww]}{8} ==$$
$$-\frac{7\,CW[tttw]}{12} - \frac{17\,CW[ttww]}{24} + \frac{2\,CW[twtw]}{3} - 4\,CW[twww] + \frac{3\,CW[wwww]}{8}$$

4. The h equation

```
Print /@ {
   1 → (t1 = J[u, BCH[α, β]]),
   2 → (t2 = J[u, α]),
   3 → (t3 = J[u, β // RC[u, α]] // CC[u, -α]),
   4 → t1 ≡ t2 + t3
  };
```

$$1 \to \mathrm{CWS}\left[0,\ -\widehat{tu}-2\,\widehat{uv},\ -\frac{19\,\widehat{ttu}}{6}-\frac{10\,\widehat{tuu}}{3}+\frac{\widehat{tuv}}{4}+\frac{\widehat{tvu}}{12}+\frac{\widehat{uuv}}{3}-\frac{17\,\widehat{uvv}}{6}\right]$$

$$2 \to \mathrm{CWS}\left[\widehat{u},\ -\widehat{tu}+\frac{\widehat{uv}}{2},\ -\frac{5\,\widehat{ttu}}{3}-\frac{\widehat{tuu}}{3}+\frac{3\,\widehat{tuv}}{4}-\frac{3\,\widehat{tvu}}{4}+\frac{\widehat{uuv}}{4}-\frac{5\,\widehat{uvv}}{3}\right]$$

$$3 \to \mathrm{CWS}\left[-\widehat{u},\ -\frac{5\,\widehat{uv}}{2},\ -\frac{3\,\widehat{ttu}}{2}-3\,\widehat{tuu}-\frac{\widehat{tuv}}{2}+\frac{5\,\widehat{tvu}}{6}+\frac{\widehat{uuv}}{12}-\frac{7\,\widehat{uvv}}{6}\right]$$

$4 \to \mathrm{True}$

- h and S

```
(Plus[
    Ju[γ] // RCu[γ],
    Ju[-γ // RCu[γ]]
   ] // RCu[-γ // RCu[γ]]) @ {6}
```

```
CWS[0, 0, 0, 0, 0, 0]
```

5. The meaning(s) of RC

```
Print /@ {
   1 → α,
   2 → (t1 = α // CC[u, γ] // RC[u, -γ]),
   3 → α ≡ t1
  };
```

$$1 \to \mathrm{LS}\left[\overline{u}+2\,\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}+\frac{\overline{uv}}{2},\right.$$
$$\left.-\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{5}{3}\,\overline{t\,\overline{tv}}-\overline{t\,\overline{uv}}-\frac{1}{3}\,\overline{u\,\overline{uv}}+\frac{5}{6}\,\overline{\overline{tu}\,u}-2\,\overline{\overline{tv}\,u}+\frac{1}{2}\,\overline{\overline{tv}\,v}-\frac{11}{6}\,\overline{\overline{uv}\,v}\right]$$

$$2 \to \mathrm{LS}\left[\overline{u}+2\,\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}+\frac{\overline{uv}}{2},\right.$$
$$\left.-\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{5}{3}\,\overline{t\,\overline{tv}}-\overline{t\,\overline{uv}}-\frac{1}{3}\,\overline{u\,\overline{uv}}+\frac{5}{6}\,\overline{\overline{tu}\,u}-2\,\overline{\overline{tv}\,u}+\frac{1}{2}\,\overline{\overline{tv}\,v}-\frac{11}{6}\,\overline{\overline{uv}\,v}\right]$$

$3 \to \mathrm{True}$

```
Print /@ {
   1 → α,
   2 → (t1 = α // CC[u, γ // RC[u, γ]]),
   3 → (t2 = α // RC[u, γ]),
   4 → t1 ≡ t2
  };
```

$$1 \to \mathrm{LS}\left[\overline{u}+2\,\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}+\frac{\overline{uv}}{2},\right.$$
$$\left.-\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{5}{3}\,\overline{t\,\overline{tv}}-\overline{t\,\overline{uv}}-\frac{1}{3}\,\overline{u\,\overline{uv}}+\frac{5}{6}\,\overline{tuu}-2\,\overline{tvu}+\frac{1}{2}\,\overline{tvv}-\frac{11}{6}\,\overline{uvv}\right]$$

$$2 \to \mathrm{LS}\left[\overline{u}+2\,\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}+\frac{\overline{uv}}{2},\ -\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{5}{3}\,\overline{t\,\overline{tv}}-\overline{t\,\overline{uv}}-\frac{7}{3}\,\overline{u\,\overline{uv}}-\frac{7}{6}\,\overline{tuu}+\frac{1}{2}\,\overline{tvv}-\frac{11}{6}\,\overline{uvv}\right]$$

$$3 \to \mathrm{LS}\left[\overline{u}+2\,\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}+\frac{\overline{uv}}{2},\ -\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{5}{3}\,\overline{t\,\overline{tv}}-\overline{t\,\overline{uv}}-\frac{7}{3}\,\overline{u\,\overline{uv}}-\frac{7}{6}\,\overline{tuu}+\frac{1}{2}\,\overline{tvv}-\frac{11}{6}\,\overline{uvv}\right]$$

$$4 \to \mathrm{True}$$

6. $C_u\,C_v$ and $\mathrm{RC}_u\,\mathrm{RC}_v$

```
Print /@ {
    1 → {α, β, γ},
    2 → (t1 = γ // CCu[α // RCv[-β]] // CCv[β]),
    3 → (t2 = γ // CCv[β // RCu[-α]] // CCu[α]),
    4 → t1 ≡ t2
  };
```

$$1 \to \left\{\mathrm{LS}\left[\overline{u}+2\,\overline{v},\ -\overline{tu}+\frac{\overline{tv}}{2}+\frac{\overline{uv}}{2},\right.\right.$$
$$\left.-\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{5}{3}\,\overline{t\,\overline{tv}}-\overline{t\,\overline{uv}}-\frac{1}{3}\,\overline{u\,\overline{uv}}+\frac{5}{6}\,\overline{tuu}-2\,\overline{tvu}+\frac{1}{2}\,\overline{tvv}-\frac{11}{6}\,\overline{uvv}\right],$$
$$\mathrm{LS}\left[-\overline{u},\ -2\,\overline{tv}+\frac{\overline{uv}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{tu}}-\frac{1}{6}\,\overline{t\,\overline{tv}}-\frac{1}{3}\,\overline{t\,\overline{uv}}-\frac{5}{6}\,\overline{u\,\overline{uv}}+2\,\overline{tuu}-2\,\overline{tvv}+\frac{11}{6}\,\overline{uvv}\right],$$
$$\left.\mathrm{LS}\left[0,\ -2\,\overline{tu}+2\,\overline{tv}+2\,\overline{uv},\ -\frac{2}{3}\,\overline{t\,\overline{tu}}-\frac{3}{2}\,\overline{t\,\overline{tv}}-\frac{1}{6}\,\overline{t\,\overline{uv}}-2\,\overline{tuu}+\frac{1}{6}\,\overline{tvu}-2\,\overline{tvv}-\frac{5}{6}\,\overline{uvv}\right]\right\}$$

$$2 \to \mathrm{LS}\left[0,\ -2\,\overline{tu}+2\,\overline{tv}+2\,\overline{uv},\right.$$
$$\left.-\frac{2}{3}\,\overline{t\,\overline{tu}}-\frac{3}{2}\,\overline{t\,\overline{tv}}+\frac{11}{6}\,\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}-2\,\overline{tuu}+\frac{1}{6}\,\overline{tvu}-2\,\overline{tvv}-\frac{29}{6}\,\overline{uvv}\right]$$

$$3 \to \mathrm{LS}\left[0,\ -2\,\overline{tu}+2\,\overline{tv}+2\,\overline{uv},\right.$$
$$\left.-\frac{2}{3}\,\overline{t\,\overline{tu}}-\frac{3}{2}\,\overline{t\,\overline{tv}}+\frac{11}{6}\,\overline{t\,\overline{uv}}-2\,\overline{u\,\overline{uv}}-2\,\overline{tuu}+\frac{1}{6}\,\overline{tvu}-2\,\overline{tvv}-\frac{29}{6}\,\overline{uvv}\right]$$

$$4 \to \mathrm{True}$$

```
Print /@ {
    1 → {α, β, γ},
    2 → (t1 = γ // RCu[α] // RCv[β // RCu[α]]),
    3 → (t2 = γ // RCv[β] // RCu[α // RCv[β]]),
    4 → t1 ≡ t2
  };
```

$$1 \to \left\{ LS\left[\overline{u} + 2\,\overline{v},\ -\overline{tu} + \frac{\overline{tv}}{2} + \frac{\overline{uv}}{2},\right.\right.$$
$$\left. -\frac{5}{3}\,\overline{t\,\overline{tu}} + \frac{5}{3}\,\overline{t\,\overline{tv}} - \overline{t\,\overline{uv}} - \frac{1}{3}\,\overline{u\,\overline{uv}} + \frac{5}{6}\,\overline{\overline{tu}\,u} - 2\,\overline{\overline{tv}\,u} + \frac{1}{2}\,\overline{\overline{tv}\,v} - \frac{11}{6}\,\overline{\overline{uv}\,v} \right],$$
$$LS\left[-\overline{u},\ -2\,\overline{tv} + \frac{\overline{uv}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{tu}} - \frac{1}{6}\,\overline{t\,\overline{tv}} - \frac{1}{3}\,\overline{t\,\overline{uv}} - \frac{5}{6}\,\overline{u\,\overline{uv}} + 2\,\overline{\overline{tu}\,u} - 2\,\overline{\overline{tv}\,v} + \frac{11}{6}\,\overline{\overline{uv}\,v} \right],$$
$$\left. LS\left[0,\ -2\,\overline{tu} + 2\,\overline{tv} + 2\,\overline{uv},\ -\frac{2}{3}\,\overline{t\,\overline{tu}} - \frac{3}{2}\,\overline{t\,\overline{tv}} - \frac{1}{6}\,\overline{t\,\overline{uv}} - 2\,\overline{\overline{tu}\,u} + \frac{1}{6}\,\overline{\overline{tv}\,u} - 2\,\overline{\overline{tv}\,v} - \frac{5}{6}\,\overline{\overline{uv}\,v} \right] \right\}$$

$$2 \to LS\left[0,\ -2\,\overline{tu} + 2\,\overline{tv} + 2\,\overline{uv},\right.$$
$$\left. -\frac{2}{3}\,\overline{t\,\overline{tu}} - \frac{3}{2}\,\overline{t\,\overline{tv}} + \frac{11}{6}\,\overline{t\,\overline{uv}} - 2\,\overline{u\,\overline{uv}} - 2\,\overline{\overline{tu}\,u} + \frac{1}{6}\,\overline{\overline{tv}\,u} - 2\,\overline{\overline{tv}\,v} - \frac{29}{6}\,\overline{\overline{uv}\,v} \right]$$

$$3 \to LS\left[0,\ -2\,\overline{tu} + 2\,\overline{tv} + 2\,\overline{uv},\right.$$
$$\left. -\frac{2}{3}\,\overline{t\,\overline{tu}} - \frac{3}{2}\,\overline{t\,\overline{tv}} + \frac{11}{6}\,\overline{t\,\overline{uv}} - 2\,\overline{u\,\overline{uv}} - 2\,\overline{\overline{tu}\,u} + \frac{1}{6}\,\overline{\overline{tv}\,u} - 2\,\overline{\overline{tv}\,v} - \frac{29}{6}\,\overline{\overline{uv}\,v} \right]$$

$$4 \to \text{True}$$

7.

8.

9.

10.

11. div property uv

```
Print /@ {
    0 → {α, β},
    1 → (t1 = divu[α] // adv[β]),
    2 → (t2 = divv[β] // adu[α]),
    3 → (t3 = MakeCWSeries[0]),
    4 → (t4 = divu[α // adv[β]]),
    5 → (t5 = divv[β // adu[α]]),
    6 → t1 - t2 ≡ t3 + t4 - t5
  };
```

$$0 \to \left\{ \mathrm{LS}\left[\overline{u} + 2\,\overline{v},\ -\overline{t\,u} + \frac{\overline{t\,v}}{2} + \frac{\overline{u\,v}}{2},\right.\right.$$
$$\left. -\frac{5}{3}\,\overline{t\,\overline{t\,u}} + \frac{5}{3}\,\overline{t\,\overline{t\,v}} - \overline{t\,\overline{u\,v}} - \frac{1}{3}\,\overline{u\,\overline{u\,v}} + \frac{5}{6}\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,u} + \frac{1}{2}\,\overline{\overline{t\,v}\,v} - \frac{11}{6}\,\overline{\overline{u\,v}\,v}\right],$$
$$\left. \mathrm{LS}\left[-\overline{u},\ -2\,\overline{t\,v} + \frac{\overline{u\,v}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{t\,u}} - \frac{1}{6}\,\overline{t\,\overline{t\,v}} - \frac{1}{3}\,\overline{t\,\overline{u\,v}} - \frac{5}{6}\,\overline{u\,\overline{u\,v}} + 2\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,v} + \frac{11}{6}\,\overline{\overline{u\,v}\,v}\right]\right\}$$

$$1 \to \mathrm{CWS}[0,\ 0,\ 0]$$

$$2 \to \mathrm{CWS}[0,\ 0,\ 0]$$

$$3 \to \mathrm{CWS}[0,\ 0,\ 0]$$

$$4 \to \mathrm{CWS}\left[0,\ 2\,\widehat{u\,v},\ \frac{\widehat{t\,v\,u}}{2} + \frac{\widehat{u\,u\,v}}{2} + \widehat{u\,v\,v}\right]$$

$$5 \to \mathrm{CWS}\left[0,\ 2\,\widehat{u\,v},\ \frac{\widehat{t\,v\,u}}{2} + \frac{\widehat{u\,u\,v}}{2} + \widehat{u\,v\,v}\right]$$

$$6 \to \mathrm{True}$$

12. div property uu

```
Print /@ {
   0 → {α, β},
   1 → (t1 = divu[α] // adu[β]),
   2 → (t2 = divu[β] // adu[α]),
   3 → (t3 = divu[b[α, β]]),
   4 → (t4 = divu[α // adu[β]]),
   5 → (t5 = divu[β // adu[α]]),
   6 → t1 - t2 ≡ t3 + t4 - t5
  };
```

$$0 \to \left\{ \mathrm{LS}\left[\overline{u} + 2\,\overline{v},\ -\overline{t\,u} + \frac{\overline{t\,v}}{2} + \frac{\overline{u\,v}}{2},\right.\right.$$
$$\left. -\frac{5}{3}\,\overline{t\,\overline{t\,u}} + \frac{5}{3}\,\overline{t\,\overline{t\,v}} - \overline{t\,\overline{u\,v}} - \frac{1}{3}\,\overline{u\,\overline{u\,v}} + \frac{5}{6}\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,u} + \frac{1}{2}\,\overline{\overline{t\,v}\,v} - \frac{11}{6}\,\overline{\overline{u\,v}\,v}\right],$$
$$\left. \mathrm{LS}\left[-\overline{u},\ -2\,\overline{t\,v} + \frac{\overline{u\,v}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{t\,u}} - \frac{1}{6}\,\overline{t\,\overline{t\,v}} - \frac{1}{3}\,\overline{t\,\overline{u\,v}} - \frac{5}{6}\,\overline{u\,\overline{u\,v}} + 2\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,v} + \frac{11}{6}\,\overline{\overline{u\,v}\,v}\right]\right\}$$

$$1 \to \mathrm{CWS}[0,\ 0,\ 0]$$

$$2 \to \mathrm{CWS}[0,\ 0,\ 0]$$

$$3 \to \mathrm{CWS}\left[0,\ -2\,\widehat{u\,v},\ -\widehat{t\,u\,u} - \frac{3\,\widehat{t\,u\,v}}{2} + \frac{3\,\widehat{t\,v\,u}}{2} - \widehat{u\,u\,v} - \widehat{u\,v\,v}\right]$$

$$4 \to \mathrm{CWS}\left[0,\ 0,\ 2\,\widehat{t\,u\,v} - 2\,\widehat{t\,v\,u} + \frac{\widehat{u\,u\,v}}{2}\right]$$

$$5 \to \mathrm{CWS}\left[0,\ -2\,\widehat{u\,v},\ -\widehat{t\,u\,u} + \frac{\widehat{t\,u\,v}}{2} - \frac{\widehat{t\,v\,u}}{2} - \frac{\widehat{u\,u\,v}}{2} - \widehat{u\,v\,v}\right]$$

$$6 \to \mathrm{True}$$

13.

14.

15. The relation with tder, 1

```
Print /@ {
   0 → γ,
   1 → (t1 = γ // DerivationExp[ad_u[γ]]),
   2 → (t2 = β // DerivationExp[ad_u[γ]]),
   3 → (t3 = β // CC[u, γ // DerivationExp[ad_u[γ]]]),
   4 → t2 ≡ t3
  };
```

$$0 \to \mathrm{LS}\left[-2\,\overline{t},\ -\frac{\overline{tu}}{2}+\frac{3\,\overline{tv}}{2}+\frac{3\,\overline{uv}}{2},\ 2\,\overline{t\,\overline{tv}}+\frac{2}{3}\,\overline{t\,\overline{uv}}-\frac{5}{6}\,\overline{u\,\overline{uv}}-\overline{t\,\overline{uu}}-\frac{1}{3}\,\overline{tv\,\overline{u}}-\frac{5}{3}\,\overline{tv\,\overline{v}}+\frac{4}{3}\,\overline{uv\,\overline{v}}\right]$$

$$1 \to \mathrm{LS}\left[-2\,\overline{t},\ -\frac{\overline{tu}}{2}+\frac{3\,\overline{tv}}{2}+\frac{3\,\overline{uv}}{2},\ \overline{t\,\overline{tu}}+2\,\overline{t\,\overline{tv}}-\frac{7}{3}\,\overline{t\,\overline{uv}}-\frac{5}{6}\,\overline{u\,\overline{uv}}-\overline{t\,\overline{uu}}-\frac{10}{3}\,\overline{tv\,\overline{u}}-\frac{5}{3}\,\overline{tv\,\overline{v}}+\frac{4}{3}\,\overline{uv\,\overline{v}}\right]$$

$$2 \to \mathrm{LS}\left[-\overline{t}+\overline{u}-2\,\overline{v},\ -\frac{5\,\overline{tu}}{2}+\overline{tv},\ \frac{29}{6}\,\overline{t\,\overline{tu}}+2\,\overline{t\,\overline{tv}}-\frac{1}{6}\,\overline{t\,\overline{uv}}-\frac{11}{6}\,\overline{u\,\overline{uv}}+\frac{5}{6}\,\overline{t\,\overline{uu}}+\frac{4}{3}\,\overline{tv\,\overline{u}}-\frac{1}{6}\,\overline{tv\,\overline{v}}+\frac{11}{6}\,\overline{uv\,\overline{v}}\right]$$

$$3 \to \mathrm{LS}\left[-\overline{t}+\overline{u}-2\,\overline{v},\ -\frac{5\,\overline{tu}}{2}+\overline{tv},\ \frac{29}{6}\,\overline{t\,\overline{tu}}+2\,\overline{t\,\overline{tv}}-\frac{1}{6}\,\overline{t\,\overline{uv}}-\frac{11}{6}\,\overline{u\,\overline{uv}}+\frac{5}{6}\,\overline{t\,\overline{uu}}+\frac{4}{3}\,\overline{tv\,\overline{u}}-\frac{1}{6}\,\overline{tv\,\overline{v}}+\frac{11}{6}\,\overline{uv\,\overline{v}}\right]$$

$$4 \to -\frac{59\langle tttu\rangle}{12}+\frac{31\langle tttv\rangle}{24}+\frac{\langle ttuu\rangle}{3}+\frac{7\langle ttuv\rangle}{6}+\frac{43\langle ttvu\rangle}{24}-\frac{11\langle ttvv\rangle}{6}+\frac{10\langle tutv\rangle}{3}+\frac{7\langle tuuu\rangle}{8}+\frac{9\langle tuuv\rangle}{2}+\frac{17\langle tuvu\rangle}{12}-\frac{29\langle tuvv\rangle}{6}-\frac{13\langle tvuu\rangle}{4}-\frac{97\langle tvuv\rangle}{12}-\frac{115\langle tvvu\rangle}{24}+\frac{15\langle tvvv\rangle}{8}+\frac{13\langle uuuv\rangle}{24}-\frac{7\langle uuvv\rangle}{3}-\frac{5\langle uvvv\rangle}{24}==-\frac{59\langle tttu\rangle}{12}+\frac{31\langle tttv\rangle}{24}+\frac{5\langle ttuu\rangle}{6}+\frac{7\langle ttuv\rangle}{6}+\frac{43\langle ttvu\rangle}{24}-\frac{11\langle ttvv\rangle}{6}+\frac{10\langle tutv\rangle}{3}+\frac{7\langle tuuu\rangle}{8}+\frac{9\langle tuuv\rangle}{2}-\frac{\langle tuvu\rangle}{12}-\frac{29\langle tuvv\rangle}{6}-\frac{19\langle tvuu\rangle}{4}-\frac{97\langle tvuv\rangle}{12}-\frac{115\langle tvvu\rangle}{24}+\frac{15\langle tvvv\rangle}{8}+\frac{13\langle uuuv\rangle}{24}-\frac{7\langle uuvv\rangle}{3}-\frac{5\langle uvvv\rangle}{24}$$

16.

17.

18.

19.

20. The differential of BCH

```
Print /@ {
   1 → (bch = BCH[u, v]),
   2 → (BCH[u + ϵ t, v + ϵ w] - bch)/ϵ,
   3 → (t1 = (BCH[u + ϵ t, v + ϵ w] - bch)/ϵ // adSeries[(1 - e^-ad)/ad, bch]),
   4 → (t2 = t // adSeries[(1 - e^-ad)/ad, u] // Ad[-v]),
   5 → (t3 = w // adSeries[(1 - e^-ad)/ad, v])
  };
t1 ≡ t2 + t3
```

$$1 \to \mathrm{LS}\left[\overline{u}+\overline{v},\ \frac{\overline{uv}}{2},\ \frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{\overline{uv}v}\right]$$

$$2 \to \mathrm{LS}\left[\overline{t}+\overline{w},\ \frac{\overline{tv}}{2}+\frac{\overline{uw}}{2},\ \frac{1}{12}\overline{t\overline{uv}}+\frac{1}{12}\overline{u\overline{uw}}+\frac{1}{12}\overline{u\overline{vw}}-\frac{1}{12}\overline{\overline{tv}u}+\frac{1}{12}\overline{\overline{tv}v}+\frac{1}{6}\overline{\overline{uw}v}\right]$$

$$3 \to \mathrm{LS}\left[\overline{t}+\overline{w},\ \frac{\overline{tu}}{2}+\overline{tv}-\frac{\overline{vw}}{2},\ \frac{1}{2}\overline{t\overline{uv}}+\frac{1}{6}\overline{v\overline{vw}}+\frac{1}{6}\overline{\overline{tu}u}+\frac{1}{2}\overline{\overline{tv}u}+\frac{1}{2}\overline{\overline{tv}v}\right]$$

$$4 \to \mathrm{LS}\left[\overline{t},\ \frac{\overline{tu}}{2}+\overline{tv},\ \frac{1}{2}\overline{t\overline{uv}}+\frac{1}{6}\overline{\overline{tu}u}+\frac{1}{2}\overline{\overline{tv}u}+\frac{1}{2}\overline{\overline{tv}v}\right]$$

$$5 \to \mathrm{LS}\left[\overline{w},\ -\frac{\overline{vw}}{2},\ \frac{1}{6}\overline{v\overline{vw}}\right]$$

```
True
```

21. The differential of C

```
Print /@ {
   0 → {α, β, γ},
   1 → (t1 = ((γ // CC[u, α + ϵ β]) - (γ // CC[u, α]))/ϵ),
   2 → (t2 = γ // ad[u, adSeries[(e^ad - 1)/ad, α][β] // RC[u, -α]] // CC[u, α]),
   t1 ≡ t2
  };
```

$$0 \to \Big\{ LS\Big[\overline{u} + 2\,\overline{v},\ -\overline{t\,u} + \frac{\overline{t\,v}}{2} + \frac{\overline{u\,v}}{2},$$
$$-\frac{5}{3}\,\overline{t\,\overline{t\,u}} + \frac{5}{3}\,\overline{t\,\overline{t\,v}} - \overline{t\,\overline{u\,v}} - \frac{1}{3}\,\overline{u\,\overline{u\,v}} + \frac{5}{6}\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,u} + \frac{1}{2}\,\overline{\overline{t\,v}\,v} - \frac{11}{6}\,\overline{\overline{u\,v}\,v}\Big],$$
$$LS\Big[-\overline{u},\ -2\,\overline{t\,v} + \frac{\overline{u\,v}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{t\,u}} - \frac{1}{6}\,\overline{t\,\overline{t\,v}} - \frac{1}{3}\,\overline{t\,\overline{u\,v}} - \frac{5}{6}\,\overline{u\,\overline{u\,v}} + 2\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,v} + \frac{11}{6}\,\overline{\overline{u\,v}\,v}\Big],$$
$$LS\Big[0,\ -2\,\overline{t\,u} + 2\,\overline{t\,v} + 2\,\overline{u\,v},\ -\frac{2}{3}\,\overline{t\,\overline{t\,u}} - \frac{3}{2}\,\overline{t\,\overline{t\,v}} - \frac{1}{6}\,\overline{t\,\overline{u\,v}} - 2\,\overline{\overline{t\,u}\,u} + \frac{1}{6}\,\overline{\overline{t\,v}\,u} - 2\,\overline{\overline{t\,v}\,v} - \frac{5}{6}\,\overline{\overline{u\,v}\,v}\Big]\Big\}$$

```
1 → LS[0, 0, 0]

2 → LS[0, 0, 0]

True
```

22. The differential of RC

```
Print /@ {
   0 → {α, β, γ},
   1 → (t1 = ((γ // RC[u, α + ϵ β]) - (γ // RC[u, α]))/ϵ),
   2 → (t2 = γ // RC[u, α] // ad[u, adSeries[(1 - e^-ad)/ad, α][β] // RC[u, α]]),
   t1 ≡ t2
  };
```

$$0 \to \Big\{ LS\Big[\overline{u} + 2\,\overline{v},\ -\overline{t\,u} + \frac{\overline{t\,v}}{2} + \frac{\overline{u\,v}}{2},$$
$$-\frac{5}{3}\,\overline{t\,\overline{t\,u}} + \frac{5}{3}\,\overline{t\,\overline{t\,v}} - \overline{t\,\overline{u\,v}} - \frac{1}{3}\,\overline{u\,\overline{u\,v}} + \frac{5}{6}\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,u} + \frac{1}{2}\,\overline{\overline{t\,v}\,v} - \frac{11}{6}\,\overline{\overline{u\,v}\,v}\Big],$$
$$LS\Big[-\overline{u},\ -2\,\overline{t\,v} + \frac{\overline{u\,v}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{t\,u}} - \frac{1}{6}\,\overline{t\,\overline{t\,v}} - \frac{1}{3}\,\overline{t\,\overline{u\,v}} - \frac{5}{6}\,\overline{u\,\overline{u\,v}} + 2\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,v} + \frac{11}{6}\,\overline{\overline{u\,v}\,v}\Big],$$
$$LS\Big[0,\ -2\,\overline{t\,u} + 2\,\overline{t\,v} + 2\,\overline{u\,v},\ -\frac{2}{3}\,\overline{t\,\overline{t\,u}} - \frac{3}{2}\,\overline{t\,\overline{t\,v}} - \frac{1}{6}\,\overline{t\,\overline{u\,v}} - 2\,\overline{\overline{t\,u}\,u} + \frac{1}{6}\,\overline{\overline{t\,v}\,u} - 2\,\overline{\overline{t\,v}\,v} - \frac{5}{6}\,\overline{\overline{u\,v}\,v}\Big]\Big\}$$

```
1 → LS[0, 0, 0]

2 → LS[0, 0, 0]

True
```

23. The differential of J

```
Print /@ {
    0 → {α, β},
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = ∫_0^1 (div[u, β // RC[u, s α]] // CC[u, -s α]) ds),
    3 → (t2 = ∫_0^1 (s div[u, α // RC[u, s α] //
                ad[u, β // adSeries[(1 - e^(-s ad))/(s ad), α] // RC[u, s α]]] // CC[u, -s α]) ds),
    4 → (t3 = ∫_0^1 (s div[u, α // RC[u, s α]] // ad[u, β // adSeries[(1 - e^(-s ad))/(s ad), α] //
                RC[u, s α]] // CC[u, -s α]) ds),
    t0 ≡ t1 + t2 - t3
  };
```

$$0 \to \left\{\mathrm{LS}\left[\overline{u} + 2\,\overline{v},\ -\overline{t\,u} + \frac{\overline{t\,v}}{2} + \frac{\overline{u\,v}}{2},\right.\right.$$
$$\left. -\frac{5}{3}\,\overline{t\,\overline{t\,u}} + \frac{5}{3}\,\overline{t\,\overline{t\,v}} - \overline{t\,\overline{u\,v}} - \frac{1}{3}\,\overline{u\,\overline{u\,v}} + \frac{5}{6}\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,u} + \frac{1}{2}\,\overline{\overline{t\,v}\,v} - \frac{11}{6}\,\overline{\overline{u\,v}\,v}\right],$$
$$\left.\mathrm{LS}\left[-\overline{u},\ -2\,\overline{t\,v} + \frac{\overline{u\,v}}{2},\ -\frac{3}{2}\,\overline{t\,\overline{t\,u}} - \frac{1}{6}\,\overline{t\,\overline{t\,v}} - \frac{1}{3}\,\overline{t\,\overline{u\,v}} - \frac{5}{6}\,\overline{u\,\overline{u\,v}} + 2\,\overline{\overline{t\,u}\,u} - 2\,\overline{\overline{t\,v}\,v} + \frac{11}{6}\,\overline{\overline{u\,v}\,v}\right]\right\}$$

$$1 \to \mathrm{CWS}\left[-\widehat{u},\ -\frac{3\,\widehat{u\,v}}{2},\ -\frac{3\,\widehat{t\,t\,u}}{2} - \frac{5\,\widehat{t\,u\,u}}{2} + \frac{5\,\widehat{t\,u\,v}}{4} - \frac{11\,\widehat{t\,v\,u}}{12} + \frac{3\,\widehat{u\,u\,v}}{2} + \frac{2\,\widehat{u\,v\,v}}{3}\right]$$

$$2 \to \mathrm{CWS}\left[-\widehat{u},\ -\frac{3\,\widehat{u\,v}}{2},\ -\frac{3\,\widehat{t\,t\,u}}{2} - \frac{5\,\widehat{t\,u\,u}}{2} + \frac{\widehat{t\,u\,v}}{4} + \frac{\widehat{t\,v\,u}}{12} + \frac{11\,\widehat{u\,u\,v}}{12} + \frac{2\,\widehat{u\,v\,v}}{3}\right]$$

$$3 \to \mathrm{CWS}\left[0,\ 0,\ \widehat{t\,u\,v} - \widehat{t\,v\,u} + \frac{7\,\widehat{u\,u\,v}}{12}\right]$$

$$4 \to \mathrm{CWS}[0,\ 0,\ 0]$$

True

```
Print /@ {
   0 → {α, β};
   1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
   2 → (t1 = ∫_0^1 (div[u, β // RC[u, s α]] // CC[u, -s α]) ds),
   3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
            ad[u, α // RC[u, s α]]] // CC[u, -s α]) ds),
   4 → (t3 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
          ad[u, α // RC[u, s α]] // CC[u, -s α]) ds),
   5 → (t4 = ∫_0^1 (div[u, b[α // RC[u, s α], β // adSeries[(1 - e^(-s ad))/ad, α] //
            RC[u, s α]]] // CC[u, -s α]) ds),
   t0 ≡ t1 + t2 - t3 - t4
  };
```

$$1 \to \mathrm{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2}-\frac{5\,\widehat{tuu}}{2}+\frac{5\,\widehat{tuv}}{4}-\frac{11\,\widehat{tvu}}{12}+\frac{3\,\widehat{uuv}}{2}+\frac{2\,\widehat{uvv}}{3}\right]$$

$$2 \to \mathrm{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2}-\frac{5\,\widehat{tuu}}{2}+\frac{\widehat{tuv}}{4}+\frac{\widehat{tvu}}{12}+\frac{11\,\widehat{uuv}}{12}+\frac{2\,\widehat{uvv}}{3}\right]$$

$$3 \to \mathrm{CWS}\left[0, -\widehat{uv}, -\frac{\widehat{tuu}}{2}+\frac{\widehat{tuv}}{4}-\frac{\widehat{tvu}}{4}+\frac{5\,\widehat{uuv}}{12}-\frac{7\,\widehat{uvv}}{6}\right]$$

$$4 \to \mathrm{CWS}[0, 0, 0]$$

$$5 \to \mathrm{CWS}\left[0, -\widehat{uv}, -\frac{\widehat{tuu}}{2}-\frac{3\,\widehat{tuv}}{4}+\frac{3\,\widehat{tvu}}{4}-\frac{\widehat{uuv}}{6}-\frac{7\,\widehat{uvv}}{6}\right]$$

True

```
Print /@ {
   0 → {α, β};
   1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
   2 → (t1 = ∫_0^1 (div[u, β // RC[u, s α]] // CC[u, -s α]) đs),
   3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
            ad[u, α // RC[u, s α]]] // CC[u, -s α]) đs),
   4 → (t3 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
          ad[u, α // RC[u, s α]] // CC[u, -s α]) đs),
   5 → (t4 = ∫_0^1 (div[u, b[α, β // adSeries[(1 - e^(-s ad))/ad, α]]] // RC[u, s α]] //
        CC[u, -s α]) đs),
   t0 ≡ t1 + t2 - t3 - t4
  };
```

$$1 \to \mathrm{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - \frac{5\,\widehat{tuu}}{2} + \frac{5\,\widehat{tuv}}{4} - \frac{11\,\widehat{tvu}}{12} + \frac{3\,\widehat{uuv}}{2} + \frac{2\,\widehat{uvv}}{3}\right]$$

$$2 \to \mathrm{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - \frac{5\,\widehat{tuu}}{2} + \frac{\widehat{tuv}}{4} + \frac{\widehat{tvu}}{12} + \frac{11\,\widehat{uuv}}{12} + \frac{2\,\widehat{uvv}}{3}\right]$$

$$3 \to \mathrm{CWS}\left[0, -\widehat{uv}, -\frac{\widehat{tuu}}{2} + \frac{\widehat{tuv}}{4} - \frac{\widehat{tvu}}{4} + \frac{5\,\widehat{uuv}}{12} - \frac{7\,\widehat{uvv}}{6}\right]$$

$$4 \to \mathrm{CWS}[0, 0, 0]$$

$$5 \to \mathrm{CWS}\left[0, -\widehat{uv}, -\frac{\widehat{tuu}}{2} - \frac{3\,\widehat{tuv}}{4} + \frac{3\,\widehat{tvu}}{4} - \frac{\widehat{uuv}}{6} - \frac{7\,\widehat{uvv}}{6}\right]$$

True

```
Print /@ {
    0 → {α, β};
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = ∫_0^1 (div[u, β // RC[u, s α]] // CC[u, -s α]) ds),
    3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
              ad[u, α // RC[u, s α]]] // CC[u, -s α]) ds),
    4 → (t3 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
            ad[u, α // RC[u, s α]] // CC[u, -s α]) ds),
    5 → (t4 = ∫_0^1 (div[u, β // adSeries[1 - e^(-s ad), α] // RC[u, s α]] // CC[u, -s α]) ds),
    t0 ≡ t1 + t2 - t3 - t4
  };
```

$$1 \to \text{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - \frac{5\,\widehat{tuu}}{2} + \frac{5\,\widehat{tuv}}{4} - \frac{11\,\widehat{tvu}}{12} + \frac{3\,\widehat{uuv}}{2} + \frac{2\,\widehat{uvv}}{3}\right]$$

$$2 \to \text{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - \frac{5\,\widehat{tuu}}{2} + \frac{\widehat{tuv}}{4} + \frac{\widehat{tvu}}{12} + \frac{11\,\widehat{uuv}}{12} + \frac{2\,\widehat{uvv}}{3}\right]$$

$$3 \to \text{CWS}\left[0, -\widehat{uv}, -\frac{\widehat{tuu}}{2} + \frac{\widehat{tuv}}{4} - \frac{\widehat{tvu}}{4} + \frac{5\,\widehat{uuv}}{12} - \frac{7\,\widehat{uvv}}{6}\right]$$

$$4 \to \text{CWS}[0, 0, 0]$$

$$5 \to \text{CWS}\left[0, -\widehat{uv}, -\frac{\widehat{tuu}}{2} - \frac{3\,\widehat{tuv}}{4} + \frac{3\,\widehat{tvu}}{4} - \frac{\widehat{uuv}}{6} - \frac{7\,\widehat{uvv}}{6}\right]$$

True

```
Print /@ {
   0 → {α, β};
   1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
   2 → (t1 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
            ad[u, α // RC[u, s α]]] // CC[u, -s α]) ds),
   3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
           ad[u, α // RC[u, s α]] // CC[u, -s α]) ds),
   4 → (t3 = ∫_0^1 (div[u, β // adSeries[e^(-s ad), α] // RC[u, s α]] // CC[u, -s α]) ds),
   t0 ≡ t1 - t2 + t3
  };
```

$$1 \to \mathrm{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - \frac{5\,\widehat{tuu}}{2} + \frac{5\,\widehat{tuv}}{4} - \frac{11\,\widehat{tvu}}{12} + \frac{3\,\widehat{uuv}}{2} + \frac{2\,\widehat{uvv}}{3}\right]$$

$$2 \to \mathrm{CWS}\left[0, -\widehat{uv}, -\frac{\widehat{tuu}}{2} + \frac{\widehat{tuv}}{4} - \frac{\widehat{tvu}}{4} + \frac{5\,\widehat{uuv}}{12} - \frac{7\,\widehat{uvv}}{6}\right]$$

$$3 \to \mathrm{CWS}[0, 0, 0]$$

$$4 \to \mathrm{CWS}\left[-\widehat{u}, -\frac{\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - 2\,\widehat{tuu} + \widehat{tuv} - \frac{2\,\widehat{tvu}}{3} + \frac{13\,\widehat{uuv}}{12} + \frac{11\,\widehat{uvv}}{6}\right]$$

```
True
```

```
Print /@ {
   0 → {α, β};
   1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
   2 → (t1 = ∫_0^1 (div[u, β // Ad[-s α] // RC[u, s α]] // CC[u, -s α]) ds),
   3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
            ad[u, α // RC[u, s α]]] // CC[u, -s α]) ds),
   4 → (t3 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
          ad[u, α // RC[u, s α]] // CC[u, -s α]) ds),
   t0 ≡ t1 + t2 - t3
  };
```

$$1 \to \text{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - \frac{5\,\widehat{tuu}}{2} + \frac{5\,\widehat{tuv}}{4} - \frac{11\,\widehat{tvu}}{12} + \frac{3\,\widehat{uuv}}{2} + \frac{2\,\widehat{uvv}}{3}\right]$$

$$2 \to \text{CWS}\left[-\widehat{u}, -\frac{\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - 2\,\widehat{tuu} + \widehat{tuv} - \frac{2\,\widehat{tvu}}{3} + \frac{13\,\widehat{uuv}}{12} + \frac{11\,\widehat{uvv}}{6}\right]$$

$$3 \to \text{CWS}\left[0, -\widehat{uv}, -\frac{\widehat{tuu}}{2} + \frac{\widehat{tuv}}{4} - \frac{\widehat{tvu}}{4} + \frac{5\,\widehat{uuv}}{12} - \frac{7\,\widehat{uvv}}{6}\right]$$

$$4 \to \text{CWS}[0, 0, 0]$$

True

```
Print /@ {
   0 → {α, β};
   1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
   2 → (t1 = div[u, β // adSeries[(1 - e^(-ad))/ad, α] // RC[u, α]] // CC[u, -α]),
   t0 ≡ t1
  };
```

$$1 \to \text{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - \frac{5\,\widehat{tuu}}{2} + \frac{5\,\widehat{tuv}}{4} - \frac{11\,\widehat{tvu}}{12} + \frac{3\,\widehat{uuv}}{2} + \frac{2\,\widehat{uvv}}{3}\right]$$

$$2 \to \text{CWS}\left[-\widehat{u}, -\frac{3\,\widehat{uv}}{2}, -\frac{3\,\widehat{ttu}}{2} - \frac{5\,\widehat{tuu}}{2} + \frac{5\,\widehat{tuv}}{4} - \frac{11\,\widehat{tvu}}{12} + \frac{3\,\widehat{uuv}}{2} + \frac{2\,\widehat{uvv}}{3}\right]$$

True