The associated graded of a semi-direct product

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Claim If $G \circlearrowright H$ and $G \circlearrowright H^{ab}$ is trivial, then

$$gr(G \ltimes H) \underset{v.s.}{\cong} grG \otimes grH$$

Pf Given Z_G, Z_H, set

$$Z(g,h) = Z_G(g) \otimes Z_H(h)$$

$$(h-1)(g-1) = hg - g - h + 1 =$$
$$= gh^g - g - h + 1$$
$$= (g-1)(h^g - 1) + (h^g - h)$$

$\bar{g} := g - 1$
$\overline{g^{-1}} = g^{-1} - 1 = g^{-1}(1-g)$
$= -g^{-1}\bar{g}$
$=$

$$(g^{-1}hg - h) = (1 - g^{-1}\bar{g})(1+\bar{h})(1+\bar{g}) - (1+\bar{h})$$
$$= \bar{g} - g^{-1}\bar{g} + \bar{h}\bar{g} - g^{-1}\bar{g}\bar{g} - g^{-1}\bar{g}\bar{h} - g^{-1}\bar{g}\bar{h}\bar{g}$$

$$= (1-g^{-1})\bar{g} + \ldots$$
$$= \cancel{g^{-1}}\bar{g}\bar{g} - \cancel{g^{-1}}\bar{g}\bar{g} \quad \bar{h}\bar{g} - g^{-1}\bar{g}\bar{h} - g^{-1}\bar{g}\bar{h}\bar{g}$$

$$g^{-1}h^{-1}gh - 1 = (1 - g^{-1}\bar{g})(1 - h^{-1}\bar{h})(1+\bar{g})(1+\bar{h})$$

$$= -\cancel{g^{-1}\bar{g}} - \cancel{h^{-1}\bar{h}} + \cancel{\bar{g}} + \cancel{\bar{h}} + g^{-1}\bar{g}h^{-1}\bar{h} - g^{-1}\bar{g}\bar{g} - g^{-1}\bar{g}\bar{h}$$
$$- h^{-1}\bar{h}\bar{g} - h^{-1}\bar{h}\bar{h} + \bar{g}\bar{h}$$

[illegible]

$$-h^{-1}hg - \bar{h}^{-1}h\bar{h} + \bar{g}\bar{h}$$

$$= g^{-1}\bar{g}\bar{g} + \bar{h}^{-1}\bar{h}\bar{h} + g^{-1}\bar{g}h^{-1}\bar{h} - g^{-1}\bar{g}\bar{g} - g^{-1}\bar{g}\bar{h}$$

$$-h^{-1}\bar{h}\bar{g} - \bar{h}^{-1}\bar{h}\bar{h} + \bar{g}\bar{h}$$

$$=$$

$$(h^g - h) = h\left(\prod[a_i, b_i] - 1\right) =$$