Homological Perturbation Lemma

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From http://ncatlab.org/nlab/show/homological+perturbation+theory:

Homological perturbation lemma

Let $(X,d),(Y,d)$ be chain complexes over a ring R and let $f:X\to Y$, $\nabla:Y\to X$ be chain maps, and $\Phi:X\to X$ a chain homotopy such that

$$f\nabla = 1,\quad \nabla f = 1 + d\Phi + \phi d,$$
$$f\Phi = 0, \Phi\nabla = 0, \Phi^2 = 0, \Phi d\Phi = -\Phi.$$

what Crainic calls "special deformation retract"?

Let X,Y have filtrations F^* bounded below by 0 and preserved by ∇, f, Φ and the differentials on X,Y. Suppose X has another differential d^τ with the property that

$$(d^\tau - d)F^pX \subseteq F^{p-1}X$$

for all $p\geq 0$. The **Homological Perturbation Lemma** states that Y can be given a new differential d^τ such that there is a chain equivalence $(Y,d^\tau)\to(X,d^\tau)$.

$$\begin{tikzcd}
{} \arrow[r, "d"] & X^{r-1} \arrow[r, "d"] \arrow[d, shift right, "f"'] & X^{r} \arrow[r, "d"] \arrow[l, bend left, "\Phi"] \arrow[d, shift right, "f"'] & X^{r+1} \arrow[r, "d"] \arrow[l, bend left, "\Phi"] \arrow[d, shift right, "f"'] & {} \\
{} \arrow[r, "d"'] & Y^{r-1} \arrow[r, "d"'] \arrow[u, shift right, "\nabla"'] & Y^{r} \arrow[r, "d"'] \arrow[u, shift right, "\nabla"'] & Y^{r+1} \arrow[r, "d"'] \arrow[u, shift right, "\nabla"'] & {}
\end{tikzcd}$$