Anton's class 3

November-08-13
4:11 AM

Seek: A multiplicative invariant $\{\ \} \to \ /\ 4T =: \mathcal{A}$

s.t. $Z(\ ) - Z(\ ) = $ $\qquad Z(\ )$ is the MZV associator

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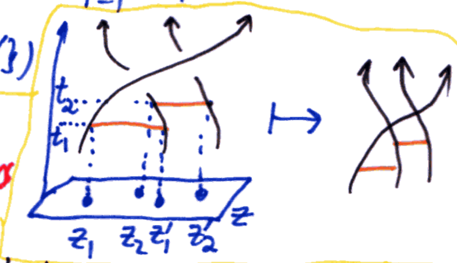
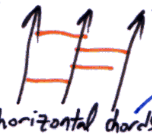
From http://www.math.toronto.edu/~drorbn/Talks/Aarhus-1305/KZ.html:

Math 1352 Algebraic Knot Theory – The Knizhnik-Zamolodchikov Connection

Theorem 1. The following is an invariant of braids in $\mathbb{R}_t \times \mathbb{C}_z$ (Fixed endpoints)

$$Z(B) = \oint_{\substack{t_1 \le \dots \le t_m \\ P = (\{z_i, z_i'\})}} \frac{D_P}{(2\pi i)^m} \bigwedge_{i=1}^m \frac{dz_i - dz_i'}{z_i - z_i'} \quad \text{in} \quad \mathcal{A}(\uparrow_n) := \left\langle t^{ij} : 1 \le i \ne j \le n \right\rangle \Big/ \begin{matrix} t^{ij} = t^{ji}; \\ [t^{ij}, t^{kl}] = 0 \\ [t^{ij}, t^{ik} + t^{jk}] = 0 \end{matrix}$$



$= \Big\langle$  $\Big\rangle \Big/ 4T$

horizontal chords.

$=$ "chord diagrams for braids".

Formal Connections & Curvature.

Let $\Omega \in \Omega^1(M, A)$ (graded unital.) with $\deg \Omega = 1$.

$\gamma : [0,1] = I \to M$ induces

$\phi : \Delta^m = \{0 \le t_1 \le \dots \le t_m \le 1\} \to M^m$.

Set $\mathrm{hol}_\gamma(\Omega) = P\exp \int_\gamma \Omega = \oint_{\Delta^m} \phi^* \Omega^m$

where $\Omega^m := \pi_1^* \Omega \wedge \dots \wedge \pi_m^* \Omega$

Theorem 2. If $F_\Omega := d\Omega + \Omega \wedge \Omega = 0$, then $\mathrm{hol}_\gamma(\Omega)$ is invariant under end-point preserving homotopies of γ.

Proof 2. Let $\Gamma : I_s \times I_t \to M$, $\Phi : I_s \times \Delta^m \to M^m$; By Stokes',

$$\int_{\Delta^m} \Phi_1^* \Omega^m - \int_{\Delta^m} \Phi_0^* \Omega^m = \int_{I \times \Delta^m} d\Phi^* \Omega^m - \int_{I \times \partial \Delta^m} \Phi^* \Omega^m =: A_m - B_m$$

Now

$$A_m = \sum_{k=1}^m (-1)^{k+1} \int_{I \times \Delta^m} \pi_1^* \Omega \wedge \dots \wedge \pi_k^* d\Omega \wedge \dots \wedge \pi_m^* \Omega$$

and

$$B_m = \int_{I \times [t_1 = 0]} \overset{\text{Fixed endpoints}}{\Phi^* \Omega^m} \pm \int_{I \times [t_m = 1]} \Phi^* \Omega^m + \sum_{k=1}^{m-1} (-1)^k \int_{I \times [t_k = t_{k+1}]} \Phi^* \Omega^m$$

$$= \sum_{k=1}^{m-1} (-1)^k \int_{I \times \Delta^{m-1}} \pi_1^* \Omega \wedge \dots \wedge \pi_k^* (\Omega \wedge \Omega) \wedge \dots \wedge \pi_{m-1}^* \Omega$$

and now $\sum A_m = \sum B_m$ by telescopic summation & $F_\Omega = 0$.

The KZ connection.

$M = \mathbb{C}^n \setminus \{\text{diagonals}\}$, $A = \mathcal{A}(\uparrow_n)$,

and $\Omega = \sum_{i<j} t^{ij} w_{ij}$ where $w_{ij} = \dfrac{dz_i - dz_j}{z_i - z_j} \underset{\text{locally}}{=} d\log(z_i - z_j)$

Compute $F_\Omega = d\Omega + \Omega \wedge \Omega$: $\quad dw_{ij} = 0$ so $d\Omega = 0$.

$$\Omega \wedge \Omega = \sum_{\substack{i<j \\ k<l}} t^{ij} t^{kl} w_{ij} \wedge w_{kl} = \underbrace{A}_{|\{ijkl\}|=2} + \underbrace{B}_{=3} + \underbrace{C}_{=4} \quad \text{where}$$

$A = C = 0$ as $[t^{ij}, t^{kl}] = 0$ if $|\{ijkl\}| = 2$ or 4 and

$$B = \sum_{\alpha < \beta < \gamma} [t^{\alpha\beta}, t^{\beta\gamma}] w_{\alpha\beta} \wedge w_{\beta\gamma} + \text{cyclic perms}$$

Note: by 4T, $[t^{\alpha\beta}, t^{\beta\gamma}] = \gamma^{\alpha\beta\gamma}$ =

$$= \sum_{\alpha < \beta < \gamma} \gamma^{\alpha\beta\gamma} \underbrace{(w_{\alpha\beta} \wedge w_{\beta\gamma} + \text{cyc perms})}_{=0 \text{ by "Arnold's identity"}} = 0$$

Proof of 1

Simply take in theorem 2, γ = the braid and Ω = the KZ connection.

Dror Bar-Natan, Feb 13, 2007

$(PaB,\ S : PaB \longrightarrow PaP,\ d_i,\ s_i)$