Anton's class 2 Based on http://drorbn.net/bbs/show?shot=Alekseev-130916-042803.jpg

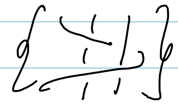
October-31-13
6:40 AM

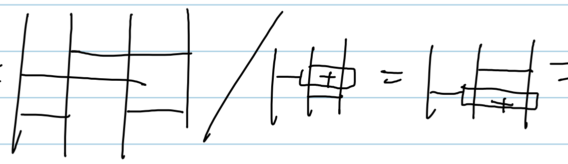
On board:

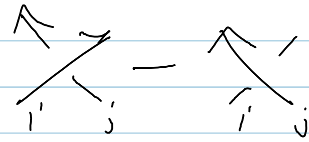
A. $G \longrightarrow \left\{ \sum a_i g_i : \begin{matrix} a_i \in \mathbb{Q} \\ g_i \in G \end{matrix} \right\} = \mathbb{Q}G \supset I = \left\{ \sum a_i g_i : \sum a_i = 0 \right\}$

Seek multiplicative $Z : \underset{(\mathbb{Q}G)}{G} \longrightarrow A := \bigoplus_{m \geq 0} I^m / I^{m+1}$

s.t. if $\gamma \in I^m$, $Z(\gamma) = (0, \ldots 0, \gamma / I^{m+1}, *, *, \ldots)$

B. $PB_n := \pi_1(C^n \setminus \text{diagonals}) = \{$  $\}$

$A_n =$  $= \langle t^{ij} = t^{ji} \rangle \Big/ \begin{matrix} [t^{ij}, t^{kl}] = 0 \\ [t^{ij} + t^{ik}, t^{jk}] = 0 \end{matrix}$

$\pi : A_n \to A_n$ by  $\longrightarrow$ 

surjective!

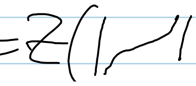
Enough to construct $\tilde{Z} : PB_n \longrightarrow A_n$ s.t.

1. $\tilde{Z}$ is "filtered": $\tilde{Z}(I^m) \subset$ degrees $\geq m$
2. If $D \in A_n$, $\operatorname{gr}\tilde{Z}(\pi(D)) = D \subset g_m A_n$ "property U"

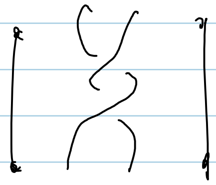
$$\begin{tikzcd} & A_n \arrow[d, "\pi"'] \\ PB_n \arrow[ur, "\tilde{Z}"] \arrow[r, dashed, "Z"'] & A_n = \bigoplus I^m / I^{m+1} \arrow[u, bend right, "\operatorname{gr}\tilde{Z}"'] \end{tikzcd}$$

.... we'll call $\tilde{Z}$ Z.

Show the Kon/KZ formula.

Claim It is enough to compute $\Phi = Z($  $)$

Claim It is enough to compute $\Phi = Z(\text{/}\!\!\!\diagup\text{/})$, $R = Z(\times)$

1.  = ⋯

2. Compute R. $R = e^{\frac{1}{2}H}$

3. Compute Φ.

$$\Phi \sim \sum_{\substack{w \in \{A,B\}^m \\ m=0}}^{\infty} I_w\, w$$

$$I_{BABBAA} = \ldots$$

Proof of invariance:

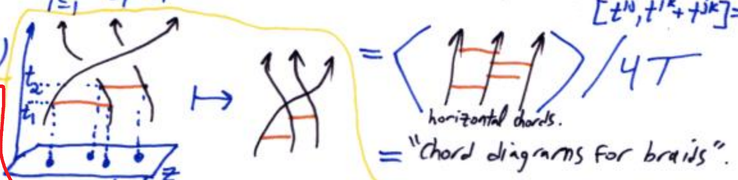
Also done: why (R, Φ) is "enough"

From http://www.math.toronto.edu/~drorbn/Talks/Aarhus-1305/KZ.html:

Math 1352 Algebraic Knot Theory – The Knizhnik-Zamolodchikov Connection

Theorem 1. The following is an invariant of braids in $\mathbb{R}_t \times \mathbb{C}_z$ (Fixed endpoints)

$$Z(B) = \sum_{\substack{t_1 \le \ldots \le t_m \\ P = (\{z_i, z_i'\})}} \frac{D_P}{(2\pi i)^m} \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \quad \text{in} \quad \mathcal{A}(\uparrow_n) := \left\langle t^{ij} : 1 \le i \ne j \le n \right\rangle \Big/ \begin{matrix} t^{ij} = t^{ji}; \\ [t^{ij}, t^{kl}] = 0 \\ [t^{ij}, t^{ik} + t^{jk}] = 0 \end{matrix}$$



done line

Formal Connections & Curvature.

Let $\Omega \in \Omega^1(M, A)$ (graded unital) with $\deg \Omega = 1$.
$\gamma : [0,1] = I \to M$ induces
$\phi : \Delta^m = \{0 \le t_1 \le \ldots \le t_m \le 1\} \to M^m$.
Set $hol_\gamma(\Omega) = P\exp \int_\gamma \Omega = \sum \int_{\Delta^m} \phi^* \Omega^m$

where $\Omega^m := \pi_1^* \Omega \wedge \ldots \wedge \pi_m^* \Omega$

Theorem 2. If $F_\Omega := d\Omega + \Omega \wedge \Omega = 0$, then $hol_\gamma(\Omega)$ is invariant under end-point preserving homotopies of γ.

Proof 2. Let $\Gamma : I_s \times I_t \to M$, $\Phi : I_s \times \Delta^m \to M^m$;
By Stokes',

$$\int_{\Delta^m} \Phi_1^* \Omega^m - \int_{\Delta^m} \Phi_0^* \Omega^m = \int_{I \times \Delta^m} d\Phi^* \Omega^m - \int_{I \times \partial \Delta^m} \Phi^* \Omega^m =: A_m - B_m$$

Now

$$A_m = \sum_{k=1}^{m} (-1)^{k+1} \int_{I \times \Delta^m} \pi_1^* \Omega \wedge \ldots \wedge \pi_k^* d\Omega \wedge \ldots \wedge \pi_m^* \Omega$$

and

$$B_m = \underbrace{\int_{I \times [t_1 = 0]} \Phi^* \Omega^m \pm \int_{I \times [t_m = 1]} \Phi^* \Omega^m}_{=0 \text{ (Fixed endpoints)}} + \sum_{k=1}^{m-1} (-1)^k \int_{I \times [t_k = t_{k+1}]} \Phi^* \Omega^m$$

$$= \sum_{k=1}^{m-1} (-1)^k \int_{I \times \Delta^{m-1}} \pi_1^* \Omega \wedge \ldots \wedge \pi_k^* (\Omega \wedge \Omega) \wedge \ldots \wedge \pi_{m-1}^* \Omega$$

and now $\sum A_m = \sum B_m$ by telescopic summation & $F_\Omega = 0$.

The KZ connection.

$M = \mathbb{C}^n \setminus \{\text{diagonals}\}$, $A = \mathcal{A}(\uparrow_n)$,

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and $\Omega=\sum_{i<j} t^{ij}w_{ij}$ where $w_{ij}=\frac{dz_i-dz_j}{z_i-z_j} \underset{\uparrow \text{locally}}{=} d\log(z_i-z_j)$

Compute $F_\Omega=d\Omega+\Omega\wedge\Omega$: $dw_{ij}=0$ so $d\Omega=0$.

$\Omega\wedge\Omega=\sum_{\substack{i<j\\k<l}} t^{ij}t^{kl}w_{ij}\wedge w_{kl}=\underbrace{A}_{|\{ijkl\}|=2}+\underbrace{B}_{=3}+\underbrace{C}_{=4}$ where

$A=C=0$ as $[t^{ij},t^{kl}]=0$ if $|\{ijkl\}|=2$ or 4 and

$B=\sum_{\alpha<\beta<\gamma}[t^{\alpha\beta},t^{\beta\gamma}]w_{\alpha\beta}\wedge w_{\beta\gamma}$ + cyclic perms

$\boxed{\text{Note: by 4T, } [t^{\alpha\beta},t^{\beta\gamma}]=Y^{\alpha\beta\gamma}=}$ 

$=\sum_{\alpha<\beta<\gamma} Y^{\alpha\beta\gamma}(w_{\alpha\beta}\wedge w_{\beta\gamma}+\text{cyc perms})=0$

$=0$ by "Arnold's identity"

Proof of 1

Simply take in theorem 2,

$Y=$ the braid

and

$\Omega=$ the KZ connection.

Dror Bar-Natan, Feb 13, 2007