

The Kashiwara-Vergne Problem and Topology

Abstract. I will describe a general machine, a class consisting of Taylor's theorem, whose inputs are topics in topology and whose outputs are problems in algebra. There are many topics the machine can take, and many outputs it produces, but I will concentrate on just one input/output pair. When fed with a certain class of knotted 2-dimensional objects in 4-dimensional space, it outputs the Kashiwara-Vergne Problem (1978 w/KV, solved Aleksey-Murshakov 2006 w/AM, candidate Aleksaev-Torresian 2008-2012 w/AT), a problem about convolutions on Lie groups and Lie algebras.

The Kashiwara-Vergne Conjecture. There exist two series F and G in the completed free Lie algebra \widehat{FL} in generators x and y so that

$$\log(\log(G^{-1} \circ (1 - \exp(-\text{ad}_x))F)) = (1 - \exp(-\text{ad}_x))^{-1} \log(1 - \exp(-\text{ad}_y))$$

and so that with $\alpha = \log(G^{-1} \circ (1 - \exp(-\text{ad}_x))F)$

$$\text{tr}(\text{ad}_x \text{ad}_y F) = \text{tr}(\text{ad}_y \text{ad}_x G)$$

in cyclic words.

$$\frac{1}{2} \text{tr} \begin{pmatrix} \text{ad}_x & & & \\ & \text{ad}_y & & \\ & & \text{ad}_x & \\ & & & \text{ad}_y \end{pmatrix}$$

Implies the recently stated **convolution statement**: Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra.

The Machine. Let G be a group, $K = \mathbb{C}\langle G \rangle = \{\sum a_i g_i \mid a_i \in \mathbb{C}, g_i \in G\}$ its group ring, $Z = \{\sum a_i g_i \mid \sum a_i = 0\} \subset K$ its augmentation ideal. Let

$$A = \mathbb{C}\langle K \rangle = \bigoplus_{n \geq 0} \mathbb{C}\langle K \rangle^n / I^{n+1}$$

Note that A inherits a product from G .

Definition. A linear $Z \rightarrow A$ is an "expansion" if for any $p \in \mathbb{Z}^n$, $Z^p = (p_1, \dots, p_n) \in \mathbb{Z}^n$, and a "homomorphic expansion" if in addition it preserves the product.

Example. Let $K = \mathbb{C}\langle \mathbb{R}^n \rangle$ and $Z = \{f \mid f(0) = 0\}$. Then $Z^n = \{f \mid f \text{ vanishes like } |x|^n\}$ in $\mathbb{C}\langle \mathbb{R}^n \rangle$ degree n homogeneous polynomials and A [power series]. The Taylor series is a homomorphic expansion!

Just for Fun

$K = \mathbb{C}\langle \mathbb{R}^n \rangle = \mathbb{C}\langle \mathbb{R}^n \rangle \oplus \mathbb{C}\langle \mathbb{R}^n \rangle \oplus \dots$ (The set of all \mathbb{R}^n variations of \mathbb{R}^n)

Group Rotation Adjoin

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An expansion Z is a collection of "progressive scalar" elements

$\mathbb{C}\langle \mathbb{R}^n \rangle = \mathbb{C}\langle \mathbb{R}^n \rangle \oplus \mathbb{C}\langle \mathbb{R}^n \rangle \oplus \mathbb{C}\langle \mathbb{R}^n \rangle \oplus \dots$

Theorem (with Zeuzanna Danesi, w/WKO). Finding a homomorphic expansion for $\mathbb{C}\langle K \rangle$ is equivalent to solving the Kashiwara-Vergne problem. There is a bijection between the set of homomorphic expansions for $\mathbb{C}\langle K \rangle$ and the set of solutions of the Kashiwara-Vergne problem. **This is the tip of an iceberg.**

4D Knots.

Lie Generators.

The Double Inflation Procedure.

WKO.

Proof of Relating A1

Daneso 2013

"God created the knots, all else in topology is the work of mortals."

Example for convolution

[www.setlas.org](http://setlas.org)

The Kashiwara-Vergne Problem

I should refresh this story

Some explanation of what a PA is could be useful.

improve?

replaced with the oc/vc pictures.

Missing: Finding a homomorphic Z is solving a system of eqns in a graded space.
 Z is more than just a ring...