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Braids and Associators, problem set 6 — by Dror Bar-Natan

Online: `http://drorbn.net/AcademicPensieve/2013-10/MZV_ex6.pdf`.

1. With $R = C^\infty(\mathbb{R}^n)$ and $I = \{f \in R \colon f(0) = 0\}$, find the set $\mathcal{Z}$ of all expansions $Z \colon R \to A := \operatorname{gr} R = \hat{\bigoplus} I^m/I^{m+1}$.

Bonus (hard). Can you find an algebraic condition that characterises the Taylor expansion Z_T within $\mathcal{Z}$? (You may want to read question 3).

2. Find a homomorphic expansion for $\mathbb{Z}F_n$, the group ring (over the integers) of the free group on n generators? (The simplest one is known as "the Magnus expansion".

3. Let G be a group and R be a ring, let $RG = \{\sum a_i g_i \colon a_i \in R\}$ be the group ring of G with coefficients in R, and let $\Delta \colon RG \to RG \otimes_R RG$ be the R-linear extension of the map $\Delta(g) = g \otimes g$. Let $I := \{\sum a_i g_i \colon \sum a_i = 0\}$ be the augmentation ideal of RG, and let $A := \operatorname{gr} RG$.

(i) Explain how Δ induces a map $\Delta_A \colon A \to A \otimes_R A$.

(ii) Describe Δ_A in the case where $RG = \mathbb{Z}F_n$.

(iii) We say that an expansion $Z \colon RG \to A$ is co-homomorphic if $(Z \otimes Z) \circ \Delta = \Delta_A \circ Z$. Is there a co-homomorphic expansion for $\mathbb{Z}F_n$? For $\mathbb{Q}F_n$?

3. Recall that $A_n := \operatorname{gr} PB_n = \langle t^{ij} = t^{ji} \colon 1 \leq i \neq j \leq n \rangle / \mathcal{R}$, where $\mathcal{R}$ consists of the relations $[t^{ij}, t^{kl}] = 0$ when $|\{i,j,k,l\}| = 4$ and $[t^{jk}, t^{ij} + t^{ik}] = 0$ when $|\{i,j,k\}| = 3$. Show that every degree m element of A_n can be written as a linear combination of sorted elements; namely, of elements of the form $t^{i_1 j_1} t^{i_2, j_2} \cdots t^{i_m j_m}$, where $i_\alpha < j_\alpha$ for every $1 \leq \alpha \leq m$ and where $j_1 \leq j_2 \leq \cdots \leq j_m$.

(This should remind you of $PB_n = F_{n-1} \rtimes (F_{n-2} \rtimes (\ldots (F_2 \rtimes F_1) \ldots))$. Does it?)