

# PROOF OF A CONJECTURE OF KULAKOVA ET AL. RELATED TO THE $\mathfrak{sl}_2$ WEIGHT SYSTEM

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ABSTRACT. In this article, we show that a conjecture raised in [KLMR], which regards the coefficient of the highest term when we evaluate the  $\mathfrak{sl}_2$  weight system on the projection of a diagram to primitive elements, is equivalent to the Melvin-Morton-Rozansky conjecture, proven in [BG].

## 1. INTRODUCTION

In this section, we briefly recall a conjecture of [KLMR] together with the relevant terminologies. A more complete treatment can be found in [KLMR]. Given a chord diagram  $D$  with  $m$  chords, its *labelled intersection graph*  $\Gamma(D)$  is the simple graph whose vertices are the chords of  $D$ , ~~numbered~~ from 1 to  $m$ , and two vertices are connected by an edge if the two corresponding chords intersect.

Following [KLMR], by orienting the chords of  $D$  arbitrarily, we can turn  $\Gamma(D)$  into an oriented graph as follows. Given two intersecting oriented chords  $a$  and  $b$ , the edge connecting  $a$  and  $b$  goes from  $a$  to  $b$  if the beginning of the chord  $b$  belongs to the arc of the outer circle of  $D$  which starts at the tail of  $a$  and goes in the positive (counter-clockwise) direction to the head of  $a$  (see Figure 1). We also have another description of the orientation. Given two intersecting oriented chords  $a$  and  $b$ , we look at the smaller arc of the outer circle of  $D$  that contains the tails of  $a$  and  $b$ . Then we orient the edge connecting  $a$  and  $b$  from  $a$  to  $b$  if we go from the tail of  $a$  to the tail of  $b$  along the smaller arc in the counter-clockwise direction. The reader should check that the two definitions of orientation are equivalent.

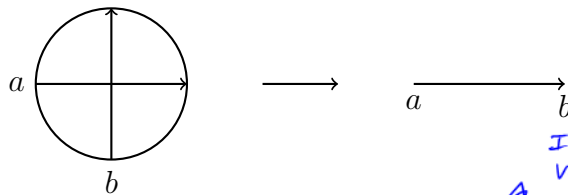


FIGURE 1. Orienting  $\Gamma(D)$

Consider a circuit of even length  $l = 2k$  in the oriented graph  $\Gamma(D)$ . By a *circuit* we mean a closed path in  $\Gamma(D)$  with no repeated edges ~~and vertices except for the first and last vertices~~. Choose an arbitrary orientation of the circuit. For each edge, we assign a weight  $+1$  if the orientation of the edge coincides with the orientation of the circuit and  $-1$  otherwise. The *sign* of a circuit is the product of the weights over all the edges in the circuit. We say that a circuit is *positively oriented* if its sign is positive and *negatively oriented* if its sign is

labelled

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I think repeated vertices are allowed. Please confirm!

you don't need to say that - there is no first and last vertices in a closed path.

negative. We define

$$R_k(D) := \sum_s \text{sign}(s),$$

where the sum is over all (un-oriented) circuits  $s$  in  $\Gamma(D)$  of length  $2k$ .

It is well-known that given a Lie algebra  $\mathfrak{g}$  equipped with an ad-invariant non-degenerate bilinear form, we can construct a weight system  $w_{\mathfrak{g}}$  with values in the center  $ZU((\mathfrak{g}))$  of the universal enveloping algebra  $U(\mathfrak{g})$  (see, for instance [CDM, Section 6]). In the case of the Lie algebra  $\mathfrak{sl}_2$ , we obtain a weight system with values in the ring  $\mathbb{C}[c]$  of polynomials in a single variable  $c$ , where  $c$  is the Casimir element of the Lie algebra  $\mathfrak{sl}_2$ . Note that the Casimir element  $c$  also depends on the choice of a bilinear form. For the case of  $\mathfrak{sl}_2$ , an ad-invariant non-degenerate bilinear form is given by *the*

$$\langle x, y \rangle = \text{Tr}(\rho(x)\rho(y)), \quad x, y \in \mathfrak{sl}_2,$$

where  $\rho: \mathfrak{sl}_2 \rightarrow \mathfrak{gl}_2$  is the standard representation of  $\mathfrak{sl}_2$ . Since  $\mathfrak{sl}_2$  is simple, any invariant form is of the form  $\lambda\langle \cdot, \cdot \rangle$  for some constant  $\lambda$ . If we let  $c_{\lambda}$  be the corresponding Casimir element and  $c = c_1$ , then  $c_{\lambda} = c/\lambda$ . If  $D$  is a chord diagram with  $n$  chords, it is known that

$$w_{\mathfrak{sl}_2}(D) = c^n + a_{n-1}c^{n-1} + \dots + a_1c$$

and the weight system corresponding to  $\lambda\langle \cdot, \cdot \rangle$  is

$$w_{\mathfrak{sl}_2, \lambda}(D) = c_{\lambda}^n + a_{n-1, \lambda}c_{\lambda}^{n-1} + \dots + a_{1, \lambda}c_{\lambda}.$$

Therefore the relationship between these two weight systems is given by

*Following [L], we now* 
$$w_{\mathfrak{sl}_2, \lambda}(D) = \frac{1}{\lambda^n} w_{\mathfrak{sl}_2}(D)|_{c=\lambda c_{\lambda}}.$$

~~Now we~~ define a map which sends a chord diagram into the set of primitive elements in the space of chord diagrams. Let  $D$  be a chord diagram with  $n$  chords,  $V = V(D)$  its set of chords. Then the map  $\pi_n$  from the space of chord diagrams to its primitive elements is given by

$$\pi_n(D) = D - 1! \sum_{V=V_1 \sqcup V_2} D|_{V_1} \cdot D|_{V_2} + 2! \sum_{V=V_1 \sqcup V_2 \sqcup V_3} D|_{V_1} \cdot D|_{V_2} \cdot D|_{V_3} - \dots,$$

where sums are taken over all unordered disjoint partitions of  $V$  into non-empty subsets and  $D|_{V_i}$  denotes  $D$  with only chords from  $V_i$  and multiplication is the usual multiplication in the space of chord diagrams. If we change unordered partitions to ordered ones, we obtain

$$(1) \quad \pi_n(D) = D - \frac{1}{2} \sum_{V=V_1 \sqcup V_2} D|_{V_1} \cdot D|_{V_2} + \frac{1}{3} \sum_{V=V_1 \sqcup V_2 \sqcup V_3} D|_{V_1} \cdot D|_{V_2} \cdot D|_{V_3} - \dots.$$

It is shown (see [L]) that  $\pi_n(D)$  is indeed a primitive element. We are ready to state the conjecture raised in [KLMR].

**Conjecture 1.** *Let  $D$  be a chord diagrams with  $2m$  chords, and  $w_{\mathfrak{sl}_2, 2}$  be the weight system associated with  $\mathfrak{sl}_2$  and  $2\langle \cdot, \cdot \rangle$ . Then*

$$w_{\mathfrak{sl}_2, 2}(\pi_{2m}(D)) = 2R_m(D)c_2^m + \dots.$$

## 2. PROOF OF THE CONJECTURE

The conjecture is a consequence of the Melvin-Morton-Rozansky (MMR) conjecture. We recall the statement of the MMR conjecture below. Let  $J^k(q)$  be the “framing independent” colored Jones polynomial associated with the  $k$ -dimensional irreducible representation of  $\mathfrak{sl}_2$ . Set  $q = e^h$ , write  $J^k(q)$  as power series in  $h$ :

$$J^k = \sum_{n=0}^{\infty} J_n^k h^n.$$

It is known that  $J_n^k$  is given by (see [O, **Theorem 6.14**] and [CDM, **Section 11.2.3**])

$$J_n^k = \text{Tr} \left( w'_{\mathfrak{sl}_2} \Big|_{c=\frac{k^2-1}{2} \cdot I_k} \right).$$

Here  $I_k$  is the  $k \times k$  identity matrix and  $w'_{\mathfrak{sl}_2}$  is the “deframing” of the weight system  $w_{\mathfrak{sl}_2}$  (see [CDM, **Section 4.5.4**]). For any chord diagram  $D$  of degree  $n$  (modulo the framing independent relation), the value  $w'_{\mathfrak{sl}_2}(D)$  is a polynomial in  $c$  of degree at most  $\lfloor n/2 \rfloor$  (see [CDM, **Exercise 6.25**]). It follows that  $J_n^k$  is a polynomial in  $k$  of degree at most  $n + 1$ . Dividing  $J_n^k$  by  $k$  we then obtain

$$\frac{J^k}{k} = \sum_{n=0}^{\infty} \left( \sum_{0 \leq j \leq n} b_{n,j} k^j \right) h^n,$$

where  $b_{n,j}$  are Vassiliev invariants of order  $\leq n$ . We denote the highest order part of the colored Jones polynomial by

$$JJ: = \sum_{n=0}^{\infty} b_{n,n} h^n.$$

Next we recall the definition of the Alexander-Conway polynomial. The Conway polynomial  $C(t)$  can be defined by the skein relation:

- (i)  $C(\text{unknot}) = 1$ ,
- (ii)  $C(L_+) - C(L_-) = tC(L_0)$ , where  $L_+$ ,  $L_-$  and  $L_0$  are identical outside the regions consisting of a positive crossing, a negative crossing and no crossing, respectively.

The Alexander-Conway polynomial is a Vassiliev power series:

$$\tilde{C}(h): = \frac{h}{e^{h/2} - e^{-h/2}} C|_{t=e^{h/2}-e^{-h/2}} = \sum_{n=0}^{\infty} c_n h^n.$$

Now we are ready to state the MMR conjecture, which was proven in [BG].

**Theorem.** *With the notations as above, we have*

$$(2) \quad JJ(h)(K) \cdot \tilde{C}(h)(K) = 1$$

for any knot  $K$ .

The proof of the MMR conjecture found in [BG] consists of reducing the equality of Vassiliev power series to an equality of weight systems. Recall that a Vassiliev invariant  $\nu$

of order  $n$  gives us a weight system  $W_n(\nu)$  of order  $n$  by  $W_n(\nu)(D) = \nu(K_D)$ , where  $D$  is a chord diagram of degree  $n$  and  $K_D$  is a singular knot whose chord diagram is  $D$ . Let

$$W_{JJ}: = \sum_{n=0}^{\infty} W_n(b_{n,n}) \text{ and } W_C: = \sum_{n=0}^{\infty} W_n(c_n).$$

Then it is shown in [BG] that the equality (2) is equivalent to

$$W_{JJ} \cdot W_C = \mathbf{1}.$$

Here  $\mathbf{1}$  denotes the weight system that takes value 1 on the empty chord diagram and 0 otherwise. Recall also that the product of two weight systems is given by

$$W_1 \cdot W_2(D) = m(W_1 \otimes W_2)(\Delta(D)),$$

where  $m$  denotes the usual multiplication in  $\mathbb{C}$  and  $\Delta$  denotes co-multiplication in the space of chord diagrams. When  $D$  is primitive, we have

$$0 = W_{JJ} \cdot W_C(D) = m(W_{JJ} \otimes W_C)(D \otimes \mathbf{1} + \mathbf{1} \otimes D) = W_{JJ}(D) + W_C(D).$$

Thus we obtain

**Lemma 1.** *If  $D$  is a chord diagram of degree  $2m$ , then*

$$W_{JJ}(\pi_{2m}(D)) = -W_C(\pi_{2m}(D)).$$

To prove conjecture 1, we need the notion of *logarithm* of a weight system (see [LZ, Chapter 6]). Let  $w$  be a weight system and suppose  $w$  can be written as  $w = \mathbf{1} + w_0$ , where  $w_0$  vanishes on chord diagrams of degree 0. Then

$$\log w: = \log(\mathbf{1} + w_0) = w_0 - \frac{1}{2}w_0^2 + \frac{1}{3}w_0^3 - \dots$$

is well-defined since for each chord diagram we only have finitely many non-zero summands.

**Lemma 2.** *Let  $w$  be a multiplicative weight system, i.e.  $w(D_1 \cdot D_2) = w(D_1)w(D_2)$ , and  $w(\text{empty chord diagram}) = 1$ . If  $D$  is a chord diagram of degree  $2m$ , then*

$$(\log w)(D) = w(\pi_{2m}(D)).$$

*Proof.* From the definition of the logarithm of a weight system we have

$$\begin{aligned} \log w &= \log(\mathbf{1} + (w - \mathbf{1})) \\ &= (w - \mathbf{1}) - \frac{1}{2}(w - \mathbf{1})^2 + \frac{1}{3}(w - \mathbf{1})^3 - \dots \end{aligned}$$

Now if  $D$  is a chord diagram, then  $(w - \mathbf{1})(\text{empty chord diagram}) = 0$  and  $(w - \mathbf{1})(D) = w(D)$  if  $D$  has degree  $> 0$ . Therefore,

$$\begin{aligned} (w - \mathbf{1})^k(D) &= \sum_{V_1 \sqcup V_2 \sqcup \dots \sqcup V_k = V(D)} w(D|_{V_1})w(D|_{V_2}) \cdots w(D|_{V_k}) \\ &= \sum_{V_1 \sqcup V_2 \sqcup \dots \sqcup V_k = V(D)} w(D|_{V_1} \cdot D|_{V_2} \cdots D|_{V_k}), \end{aligned}$$

where the sum is over ordered disjoint partition of  $V(D)$  into non-empty subsets and the last equality follows from the multiplicativity of  $w$ . Comparing with equation (1) we obtain our desired equality.  $\square$

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should be  $\mathbb{Q}$ .

It is known that the weight system  $W_C$  is multiplicative. Therefore for a chord diagram  $D$  of degree  $2m$ ,

$$(\log W_C)(D) = W_C(\pi_{2m}(D)).$$

Given an oriented circuit  $H$  in a labelled intersection graph, we define the *descent*  $d(H)$  of the circuit to be the number of label-decreases of the vertices when we go along the circuit in the specified orientation. We have the following lemma.

**Lemma 3.** *Given a chord diagram  $D$  of degree  $2m$ , we have*

$$2R_m(D) = \sum_H (-1)^{d(H)} = -(\log W_C)(D),$$

where the sum is over all oriented circuits  $H$  of length  $2m$ .

*Proof.* The second equality is proven in [BG, **Proposition 3.13**]. To prove the first equality, we show that by labeling the chords of  $D$  appropriately, the intersection graph  $\Gamma(D)$  of  $D$  has the property that the edges always go in the direction of increasing indices. To get a required labeling, we cut the outer circle of  $D$  to obtain a long chord diagram and then we label the chords as we encounter them when we go from left to right in an increasing fashion. Then it's clear that a descent will correspond to an edge with weight  $-1$ . Every circuit  $H$  will have two possible orientations  $H_+$  and  $H_-$ . However, since the circuit has even length,  $d(H_+)$  and  $d(H_-)$  have the same parity and the first equality follows.  $\square$

*Proof of Conjecture 1.* Let  $D$  be a chord diagram of degree  $2m$ , we have a chain of equalities from the above lemmas

$$2R_m(D) = \sum_H (-1)^{d(H)} = -(\log W_C)(D) = -W_C(\pi_{2m}(D)) = W_{JJ}(\pi_{2m}(D)).$$

Therefore,

$$\frac{J_{2m}^k(\pi_{2m}(D))}{k} = 2R_m(D)k^{2m} + \dots$$

Plug in  $c = (k^2 - 1)/2$  or  $k^2 = 2c + 1$  we obtain

$$w_{\mathfrak{sl}_2}(\pi_{2m}(D)) = 2^{m+1}R_m(D)c^m + \dots$$

Now we just need to do a change of variable

$$w_{\mathfrak{sl}_{2,2}}(\pi_{2m}(D)) = \frac{1}{2^{2m}} w_{\mathfrak{sl}_2}(\pi_{2m}(D))|_{c=2c_2} = 2R_m(D)c_2^m + \dots$$

and the proof is complete.  $\square$

**Remark.** Technically we need to consider  $w'_{\mathfrak{sl}_2}$  instead of  $w_{\mathfrak{sl}_2}$ . However for primitive elements, deframing does not affect the value of the highest terms (see [CDM, **Section 4.5.4**]).

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