ON A CONJECTURE RELATED TO THE $\mathfrak{sl}_2$ WEIGHT SYSTEM

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ABSTRACT. In this article, we aim to prove a conjecture raised in [4] regarding the coefficient of the highest term when we evaluate the $\mathfrak{sl}_2$ weight system on the projection of a diagram to primitive elements.

1. INTRODUCTION

In this section, we briefly recall the conjecture together with the relevant terminologies. A more complete treatment can be found in [4]. Given a chord diagram D, its *intersection graph* $\Gamma(D)$ is the simple graph whose vertices are the chords of D and two vertices are connected by an edge if the two corresponding chords intersect.

Following [4], by orienting the chords of D arbitrarily, we can turn $\Gamma(D)$ into an oriented graph as follows. Given two intersecting oriented chords A and B, the edge AB goes from A to B if the beginning of the chord B belongs to the arc of the outer circle of D which starts at the tail of A and goes in the positive (counter-clockwise) direction to the head of A (see Figure). We also have another description of the orientation. Given two intersecting oriented chords A and B, we look at the smaller arc of the outer circle of D that contains the tails of A and B. Then we orient the edge AB from A to B if we go from the tail of A to the tail of B along the smaller arc in the counter-clockwise direction. The reader should check that the two definitions of orientation are equivalent.

Now we consider a circuit of even length $l = 2k$ in the oriented graph $\Gamma(D)$. Choose an arbitrary orientation of the circuit. For each edge, we assign a weight $+1$ if the orientation of the edge coincides with the orientation of the circuit and -1 otherwise. The *sign* of a circuit is the product of the weights over all the edges in the circuit. We say that a circuit is *positively oriented* if its sign is positive and *negatively oriented* if its sign is negative. We define

$$R_k(D) \colon = \sum_c \operatorname{sign}(c),$$

where the sum is over all (un-oriented) circuits c in $\Gamma(D)$ of length $2k$.

It is well-known that given a Lie algebra $\mathfrak{g}$ equipped with an ad-invariant non-degenerate bilinear form, we can construct a weight system $w_{\mathfrak{g}}$ with values in the center $ZU((g))$ of the universal enveloping algebra $U(\mathfrak{g})$ (see, for instance [2, **Section 6**]). In the case of the Lie algebra $\mathfrak{sl}_2$, we obtain a weight system with values in the ring $\mathbb{C}[c]$ of polynomials in a single variable c, where c is the Casimir element of the Lie algebra $\mathfrak{sl}_2$. Note that the Casimir element c also depends on the choice of a bilinear form. For the case of $\mathfrak{sl}_2$, an ad-invariant non-degenerate bilinear form is given by

$$\langle x, y\rangle = \mathrm{Tr}(\rho(x)\rho(y)), \quad x, y \in \mathfrak{sl}_2,$$

where $\rho\colon \mathfrak{sl}_2 \to \mathfrak{gl}_2$ is the standard representation of $\mathfrak{sl}_2$. Since $\mathfrak{sl}_2$ is simple, any invariant form is of the form $\lambda\langle\cdot,\cdot\rangle$ for some constant λ. If we let c_λ be the corresponding Casimir element and $c = c_1$, then $c_\lambda = c/\lambda$. If D is a chord diagram with n chords, it is known that

$$w_{\mathfrak{sl}_2}(D) = c^n + a_{n-1}c^{n-1} + \cdots + a_1 c$$

and the weight system corresponding to $\lambda\langle\cdot,\cdot\rangle$ is

$$w_{\mathfrak{sl}_2,\lambda}(D) = c_\lambda^n + a_{n-1,\lambda}c_\lambda^{n-1} + \cdots + a_{1,\lambda}c_\lambda.$$

Therefore the relationship between these two weight systems is given by

$$w_{\mathfrak{sl}_2,\lambda}(D) = \frac{1}{\lambda^n}\, w_{\mathfrak{sl}_2}(D)\big|_{c=\lambda c_\lambda}.$$

Now we define a map which sends a chord diagram into the space of primitive elements. Let D be a chord diagram with n chords, $V = V(D)$ its set of chords. Then the map π_n from the space of chord diagrams to its primitive elements is given by

$$\pi_n(D) = D - 1!\sum_{V=V_1\sqcup V_2} D|_{V_1}\cdot D|_{V_2} + 2!\sum_{V=V_1\sqcup V_2\sqcup V_3} D|_{V_1}\cdot D|_{V_2}\cdot D|_{V_3} - \cdots,$$

where sums are taken over all unordered disjoint partitions of V into non-empty subsets and $D|_{V_i}$ denotes D with only chords from V_i and multiplication is the usual multiplication in the space of chord diagrams. If we change unordered partitions to ordered ones, we obtain

$$\pi_n(D) = D - \frac{1}{2}\sum_{V=V_1\sqcup V_2} D|_{V_1}\cdot D|_{V_2} + \frac{1}{3}\sum_{V=V_1\sqcup V_2\sqcup V_3} D|_{V_1}\cdot D|_{V_2}\cdot D|_{V_3} - \cdots. \tag{1}$$

It is shown (see [3]) that $\pi_n(D)$ is indeed a primitive element. We are ready to state the conjecture raised in [4].

Conjecture 1. *Let D be a chord diagrams with $2m$ chords, and $w_{\mathfrak{sl}_2,2}$ be the weight system associated with $\mathfrak{sl}_2$ and $2\langle\cdot,\cdot\rangle$. Then*

$$w_{\mathfrak{sl}_2,2}(\pi_{2m}(D)) = 2R_m(D)c_2^m + \cdots.$$

2. Proof of the conjecture

The conjecture is a consequence of the Melvin-Morton-Rozansky (MMR) conjecture. We recall the statement of the MMR conjecture below. Let $J^k(q)$ be the "framing independent" colored Jones polynomial associated with the k-dimensional irreducible representation of $\mathfrak{sl}_2$. Set $q = e^h$, write $J^k(q)$ as power series in h:

$$J^k = \sum_{n=0}^{\infty} J_n^k h^n.$$

It is known that J_n^k is given by

$$J_n^k = \mathrm{Tr}\left(w'_{\mathfrak{sl}_2}\big|_{c=\frac{k^2-1}{2}\cdot I_k} \right).$$

Here I_k is the $k \times k$ identity matrix and $w'_{\mathfrak{sl}_2}$ is the "deframing" of the weight system $w_{\mathfrak{sl}_2}$ (see [2, **Section 4.5.4**]). For any chord diagram D of degree n (modulo the framing independent relation), the value $w'_{\mathfrak{sl}_2}(D)$ is a polynomial in c of degree at most $\lfloor n/2 \rfloor$ (see [2, **Exercise 6.25**]). It follows that J_n^k is a polynomial in k of degree at most $n+1$. Dividing J_n^k by k we then obtain

$$\frac{J^k}{k} = \sum_{n=0}^{\infty} \left(\sum_{0 \le j \le n} b_{n,j} k^j \right) h^n,$$

where $b_{n,j}$ are Vassiliev invariants of order $\le n$. We denote the highest order part of the colored Jones polynomial by

$$JJ\colon = \sum_{n=0}^{\infty} b_{n,n} h^n.$$

Next we recall the definition of the Alexander-Conway polynomial. The Conway polynomial $C(t)$ can be defined by the skein relation:

(i) $C(\text{unknot}) = 1$,
(ii) $C(L_+) - C(L_-) = tC(L_0)$, where L_+, L_- and L_0 are identical outside the regions consisting of a positive crossing, a negative crossing and no crossing, respectively.

The Alexander-Conway polynomial is a Vassiliev power series:

$$\widetilde{C}(h)\colon = \frac{h}{e^{h/2} - e^{-h/2}}\, C\big|_{t=e^{h/2}-e^{-h/2}} = \sum_{n=0}^{\infty} c_n h^n.$$

Now we are ready to state the MMR conjecture, which had been proved by various people.

Theorem. *With the notations as above, we have*

$$JJ(h)(K) \cdot \widetilde{C}(h)(K) = 1 \tag{2}$$

for any knot K.

The proof of the MMR conjecture found in [1] consists of reducing the equality of Vassiliev power series to an equality of weight systems. Recall that a Vassiliev invariant ν of order n gives us a weight system $W_n(\nu)$ of order n by $W_n(\nu)(D) = \nu(K_D)$, where D is a chord diagram of degree n and K_D is a singular knot whose chord diagram is D. Let

$$W_{JJ} \colon = \sum_{n=0}^{\infty} W_n(b_{n,n}) \text{ and } W_C \colon = \sum_{n=0}^{\infty} W_n(c_n).$$

Then it is shown in [1] that the equality (2) is equivalent to

$$W_{JJ} \cdot W_C = \mathbf{1}.$$

Here $\mathbf{1}$ denotes the weight system that takes value 1 on the empty chord diagram and 0 otherwise. Recall also that the product of two weight systems is given by

$$W_1 \cdot W_2(D) = m(W_1 \otimes W_2)(\Delta(D)),$$

where m denotes multiplication and Δ denotes co-multiplication in the space of chord diagrams. When D is primitive, we have

$$0 = W_{JJ} \cdot W_C(D) = m(W_{JJ} \otimes W_C)(D \otimes 1 + 1 \otimes D) = W_{JJ}(D) + W_C(D).$$

Thus we obtain

Lemma 1. *If D is a chord diagram of degree $2m$, then*

$$W_{JJ}(\pi_{2m}(D)) = -W_C(\pi_{2m}(D)).$$

To prove conjecture 1, we need the notion of logarithm of a weight system (see [5]). Let w be a weight system and suppose w can be written as $w = \mathbf{1} + w_0$, where w_0 vanishes on chord diagrams of degree 0. Then

$$\log w \colon = \log(\mathbf{1} + w_0) = w_0 - \frac{1}{2}w_0^2 + \frac{1}{3}w_0^3 - \cdots$$

is well-defined since for each chord diagram we only have finitely many non-zero summands.

Lemma 2. *Let w be a multiplicative weight system, i.e. $w(D_1 \cdot D_2) = w(D_1)w(D_2)$, and w(empty chord diagram) $= 1$. If D is a chord diagram of degree $2m$, then*

$$(\log w)(D) = w(\pi_{2m}(D)).$$

Proof. From the definition of the logarithm of a weight system we have

$$\begin{aligned} \log w &= \log(\mathbf{1} + (w - \mathbf{1})) \\ &= (w - \mathbf{1}) - \frac{1}{2}(w - \mathbf{1})^2 + \frac{1}{3}(w - \mathbf{1})^3 - \cdots \end{aligned}$$

Now if D is a chord diagram, then $(w - \mathbf{1})$(empty chord diagram) $= 0$ and $(w - \mathbf{1})(D) = w(D)$ if D has degree > 0. Therefore,

$$(w-1)^k(D) = \sum_{V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k = V(D)} w(D|_{V_1}) w(D|_{V_2}) \cdots w(D|_{V_k})$$
$$= \sum_{V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k = V(D)} w(D|_{V_1} \cdot D|_{V_2} \cdots D|_{V_k}),$$

where the sum is over ordered disjoint partition of $V(D)$ into non-empty subsets and the last equality follows from the multiplicativity of w. Comparing with equation (1) we obtain our desired equality. $\square$

It is known that the weight system W_C is multiplicative. Therefore for a chord diagram D of degree $2m$,

$$(\log W_C)(D) = W_C(\pi_{2m}(D)).$$

Given an oriented circuit H in an oriented graph, we define the *descent* $d(H)$ of the circuit to be the number of label-decreases of the vertices when we go along the circuit in the specified orientation. We have the following lemma.

Lemma 3. *Given a chord diagram D of degree $2m$, we have*

$$2R_m(D) = \sum_H (-1)^{d(H)} = -(\log W_C)(D),$$

where the sum is over all oriented circuits H of length $2m$.

Proof. The second equality is proved in [1, **Proposition 3.13**]. To prove the first equality, we show that by labeling the chords of D appropriately, the intersection graph $\Gamma(D)$ of D has the property that the edges always go in the direction of increasing indices. To get a required labeling, we cut the outer circle of D to obtain a long chord diagram and then we label the chords as we encounter them when we go from left to right in an increasing fashion. Then it's clear that a descent will correspond to an edge with weight -1. Every circuit H will have two possible orientations H_+ and H_-. However, since the circuit has even length, $d(H_+)$ and $d(H_-)$ have the same parity and the first equality follows. $\square$

Proof of Conjecture 1. Let D be a chord diagram of degree $2m$, we have a chain of equalities from the above lemmas

$$2R_m(D) = \sum_H (-1)^{d(H)} = -(\log W_C)(D) = -W_C(\pi_{2m}(D)) = W_{JJ}(\pi_{2m}(D)).$$

Therefore,

$$\frac{J^k_{2m}(\pi_{2m}(D))}{k} = 2R_m(D)k^{2m} + \cdots .$$

Plug in $c = (k^2-1)/2$ or $k^2 = 2c+1$ we obtain

$$w_{\mathfrak{sl}_2}(\pi_{2m}(D)) = 2^{m+1} R_m(D) c^m + \cdots .$$

Now we just need to do a change of variable

$$w_{\mathfrak{sl}_2,2}(\pi_{2m}(D)) = \frac{1}{2^{2m}}\, w_{\mathfrak{sl}_2}(\pi_{2m}(D))|_{c=2c_2} = 2R_m(D)c_2^m + \cdots$$

and the proof is complete. □

Remark. Technically we need to consider $w'_{\mathfrak{sl}_2}$ instead of $w_{\mathfrak{sl}_2}$. However for primitive elements, deframing does not affect the value of the highest terms (see [2, **Section 4.5.4**]).

References

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