


Consider 2013-05/Nha Trang Post Mortem!

Put  $T_T(x)$  riddle here ✓

### Trees and Wheels and Balloons and Hoops

Dror Bar-Natan, Zurich, September 2013  
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Zurich-130919



#### 15 Minutes on Algebra

Let  $T$  be a finite set of "tail labels" and  $H$  a finite set of "head labels". Set

$$M_{1/2}(T; H) := FL(T)^H,$$

" $H$ -labeled lists of elements of the degree-completed free Lie algebra generated by  $T$ ".

$$FL(T) = \left\{ 2t_2 - \frac{1}{2}[t_1, [t_1, t_2]] + \dots \right\} / \left( \begin{array}{l} \text{anti-symmetry} \\ \text{Jacobi} \end{array} \right)$$

... with the obvious bracket.

$$M_{1/2}(u, v; x, y) = \left\{ \lambda = \left( x \rightarrow \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ x \quad y \end{array}, y \rightarrow \begin{array}{c} u \quad v \\ \diagup \quad \diagdown \\ x \quad y \end{array} \right) \dots \right\}$$

#### Operations $M_{1/2} \rightarrow M_{1/2}$ .

**Tail Multiply**  $tm_w^{uv}$  is  $\lambda \mapsto \lambda \parallel (u, v \rightarrow w)$ , satisfies "meta-associativity",  $tm_w^{uv} \parallel tm_w^{vw} = tm_w^{vw} \parallel tm_w^{uv}$ .

**Head Multiply**  $hm_z^{xy}$  is  $\lambda \mapsto (\lambda \setminus \{x, y\}) \cup (z \rightarrow \text{bch}(\lambda_x, \lambda_y))$ , where

$$\text{bch}(\alpha, \beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha, \beta]}{2} + \frac{[\alpha, [\alpha, \beta]] + [[\alpha, \beta], \beta]}{12} + \dots$$

satisfies  $\text{bch}(\text{bch}(\alpha, \beta), \gamma) = \log(e^{\alpha} e^{\beta} e^{\gamma}) = \text{bch}(\alpha, \text{bch}(\beta, \gamma))$ , and hence meta-associativity,  $hm_x^{xy} \parallel hm_x^{xz} = hm_x^{xz} \parallel hm_x^{xy}$ .

**Tail by Head Action**  $tha^{ux}$  is  $\lambda \mapsto \lambda \parallel RC_u^{\lambda_x}$ , where  $C_u^{-\gamma}: FL \rightarrow FL$  is the substitution  $u \rightarrow e^{-\gamma} u e^{\gamma}$ , or more precisely,



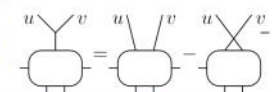
$$C_u^{-\gamma}: u \rightarrow e^{-\text{ad} \gamma}(u) = u - [\gamma, u] + \frac{1}{2}[\gamma, [\gamma, u]] - \dots,$$

and  $RC_u^{\gamma}$  is the inverse of that. Note that  $C_u^{\text{bch}(\alpha, \beta)} = C_u^{\alpha} \parallel RC_u^{\beta}$  and hence "meta  $u^{xy} = (u^x)^y$ ",

$$hm_z^{xy} \parallel tha^{uz} = tha^{uz} \parallel hm_z^{xy},$$

and  $tm_w^{uv} \parallel C_w^{\gamma} \parallel tm_w^{uv} = C_w^{\gamma} \parallel RC_w^{-\gamma} \parallel C_w^{\gamma} \parallel tm_w^{uv}$  and hence "meta  $(uv)^x = u^x v^x$ ",  $tm_w^{uv} \parallel tha^{wx} = tha^{wx} \parallel tm_w^{uv}$ .

**Wheels.** Let  $M(T; H) := M_{1/2}(T; H) \times CW(T)$ , where  $CW(T)$  is the (completed graded) vector space of cyclic words on  $T$ , or equally well, on  $FL(T)$ :

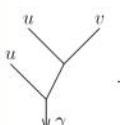
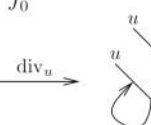
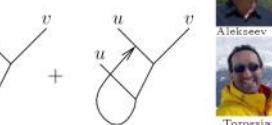




#### Operations.

On  $M(T; H)$ , define  $tm_w^{uv}$  and  $hm_z^{xy}$  as before, and  $tha^{ux}$  by adding some  $J$ -spice:

$$(\lambda; \omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) \parallel RC_u^{\lambda_x},$$

where  $J_u(\gamma) := \int_0^1 ds \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma}$ , and

#### Theorem Blue.

All blue identities still hold.

#### Merge Operation.


 $(\lambda_1; \omega_1) * (\lambda_2; \omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2).$

### 15 Minutes on Topology

$\mathcal{K}^{bh}(T; H)$

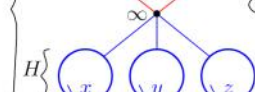
"Ribbon-knotted balloons and hoops"

$T$



balloons / tails

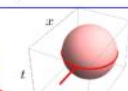
$H$




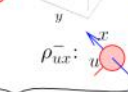
hoops / heads


$\infty$  ribbon embeddings  $\rightarrow \mathbb{R}^4 / S^4$

#### Examples.

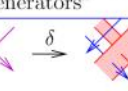
$\epsilon_x: x \rightarrow$  


$\epsilon_u: u \rightarrow$  

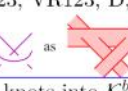
$\rho_{ux}^+: u \rightarrow x$   Shin Satoh

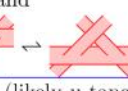
$\rho_{ux}^-: u \rightarrow x$  

"the generators"

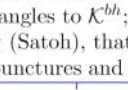
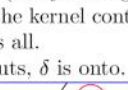
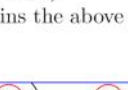
$\delta$  

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$\delta$  

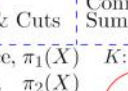
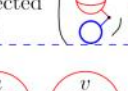
$\delta$  

satisfies R123, VR123, D, and

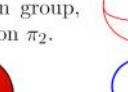
OC:  as  yet not UC: 

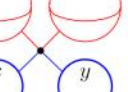
- $\delta$  injects u-knots into  $\mathcal{K}^{bh}$  (likely u-tangles too).
- $\delta$  maps v-tangles to  $\mathcal{K}^{bh}$ ; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts,  $\delta$  is onto.

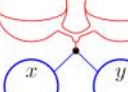
#### Operations

Punctures & Cuts  Connected Sums. 

If  $X$  is a space,  $\pi_1(X)$  is a group,  $\pi_2(X)$  is an Abelian group, and  $\pi_1$  acts on  $\pi_2$ .

$K: \begin{array}{c} u \quad v \quad w \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ x \quad y \end{array}$   $K \parallel tm_w^{uv}$ : 

$K \parallel hm_z^{xy}$ : 

$K \parallel tha^{ux}$ : 

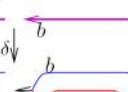
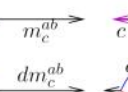

**"Meta-Group-Action"**

#### Properties

- Associativities:  $m_a^{ab} \parallel m_a^{ac} = m_b^{bc} \parallel m_a^{ab}$ , for  $m = tm, hm$ .
- " $(uv)^x = u^x v^x$ ":  $tm_w^{uv} \parallel tha^{wx} = tha^{wx} \parallel tm_w^{uv}$ .
- " $u(xy) = (u^x)^y$ ":  $hm_z^{xy} \parallel tha^{uz} = tha^{uz} \parallel hm_z^{xy}$ .

#### Tangle concatenations $\rightarrow \pi_1 \times \pi_2$ .

With  $dm_c^{ab} := tha^{ab} \parallel tm_c^{ab} \parallel hm_c^{ab}$ ,

#### Finite type invariants

make sense in the usual way, and "algebra" is (the primitive part of) "gr" of "topology".

Put the RC ops here ✓  
remove the C ops here.

remove

2013-09 Page 1

# Trees and Wheels and Balloons and Hoops: Why I Care

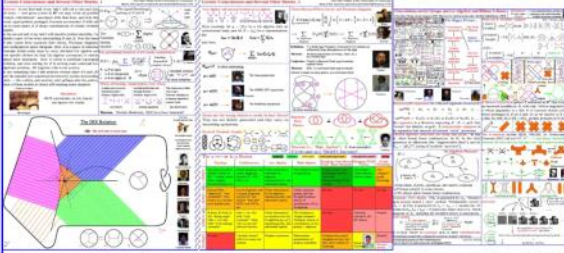
**Moral.** To construct an  $M$ -valued invariant  $\zeta$  of  $(v-)$ tangles, and nearly an invariant on  $\mathcal{K}^{bh}$ , it is enough to declare  $\zeta$  on the generators, and verify the relations that  $\delta$  satisfies.

**The Invariant  $\zeta$ .** Set  $\zeta(\epsilon_x) = (x \rightarrow 0; 0)$ ,  $\zeta(\epsilon_u) = ((); 0)$ , and

$$\zeta: \begin{array}{c} \text{tree} \\ \text{hoop} \end{array} \mapsto \begin{array}{c} \left( \begin{array}{c} u \\ \downarrow \\ x \end{array} ; 0 \right) \\ \left( - \begin{array}{c} u \\ \downarrow \\ x \end{array} ; 0 \right) \end{array}$$

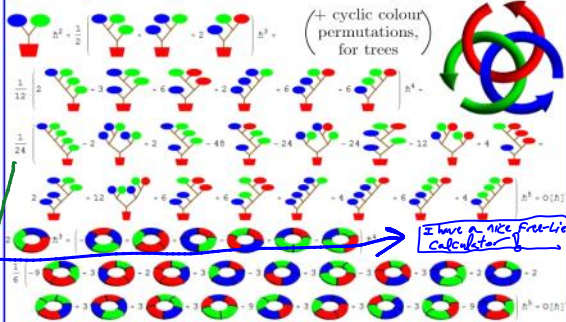
**Theorem.**  $\zeta$  is (log of) the unique homomorphic universal finite type invariant on  $\mathcal{K}^{bh}$ .  
 (... and is the tip of an iceberg)

Paper in progress with Dancso,  $\omega\epsilon\beta/wko$



See also  $\omega\epsilon\beta/tem$ ,  $\omega\epsilon\beta/bohn$ ,  $\omega\epsilon\beta/swiss$ ,  $\omega\epsilon\beta/portfolio$

**$\zeta$  is computable!**  $\zeta$  of the Borromean tangle, to degree 5:



**Tensorial Interpretation.** Let  $\mathfrak{g}$  be a finite dimensional Lie algebra (any!). Then there's  $\tau: FL(T) \rightarrow \text{Fun}(\oplus_T \mathfrak{g} \rightarrow \mathfrak{g})$  and  $\tau: CW(T) \rightarrow \text{Fun}(\oplus_T \mathfrak{g})$ . Together,  $\tau: M(T; H) \rightarrow \text{Fun}(\oplus_T \mathfrak{g} \rightarrow \oplus_H \mathfrak{g})$ , and hence

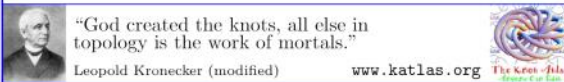
$$e^\tau: M(T; H) \rightarrow \text{Fun}(\oplus_T \mathfrak{g} \rightarrow \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

**$\zeta$  and BF Theory.** (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let  $A$  denote a  $\mathfrak{g}$ -connection on  $S^1$  with curvature  $F_A$ , and  $B$  a  $\mathfrak{g}^*$ -valued 2-form on  $S^1$ . For a hoop  $\gamma_x$ , let  $\text{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$  be the holonomy of  $A$  along  $\gamma_x$ . For a ball  $\gamma_u$ , let  $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$  be (roughly) the integral of  $B$  (transported via  $A$  to  $\infty$ ) on  $\gamma_u$ .

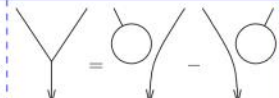
**Loose Conjecture.** For  $\gamma \in \mathcal{K}(T; H)$ ,

$$\int \mathcal{D}A \mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B)} \otimes_x \text{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is,  $\zeta$  is a complete evaluation of the BF TQFT.



The  **$\beta$  quotient** is  $M$  divided by all relations that universally hold when  $\mathfrak{g}$  is the 2D non-Abelian Lie algebra. Let  $R = \mathbb{Q}\langle\langle c_u \rangle\rangle_{u \in T}$  and  $L_\beta := R \otimes T$  with central  $R$  and with  $[u, v] = c_u v - c_v u$  for  $u, v \in T$ . Then  $FL \rightarrow L_\beta$  and  $CW \rightarrow R$ . Under this,



$\mu \rightarrow ((\lambda_x); \omega)$  with  $\lambda_x = \sum_{u \in T} \lambda_{ux} u x$ ,  $\lambda_{ux}, \omega \in R$ ,

$\text{bch}(u, v) \rightarrow \frac{c_u + c_v}{e^{c_u + c_v} - 1} \left( \frac{e^{c_u} - 1}{c_u} u + e^{c_u} \frac{e^{c_v} - 1}{c_v} v \right)$ ,

if  $\gamma = \sum \gamma_v v$  then with  $c_\gamma := \sum \gamma_v c_v$ ,

$u \parallel RC_\gamma^u = \left( 1 + c_u \gamma_u \frac{e^{c_\gamma} - 1}{c_\gamma} \right)^{-1} \left( e^{c_\gamma} u - c_u \frac{e^{c_\gamma} - 1}{c_\gamma} \sum_{v \neq u} \gamma_v v \right)$ .

$\text{div}_u \gamma = c_u \gamma_u$ , and  $J_u(\gamma) = \log \left( 1 + \frac{e^{c_\gamma} - 1}{c_\gamma} c_u \gamma_u \right)$ , so  $\zeta$  is formula-computable to all orders! **Can we simplify?**

**Repackaging.** Given  $((x \rightarrow \lambda_{ux}); \omega)$ , set  $c_x := \sum_v c_v \lambda_{vx}$ , replace  $\lambda_{ux} \rightarrow \alpha_{ux} := c_u \lambda_{ux} \frac{e^{c_x} - 1}{c_x}$  and  $\omega \rightarrow e^\omega$ , use  $t_u = e^{c_u}$ , and write  $\alpha_{ux}$  as a matrix. Get " **$\beta$  calculus**".

**$\beta$  Calculus.** Let  $\beta(T; H)$  be

$$\left\{ \begin{array}{c} \omega \\ u \\ v \\ \vdots \end{array} \middle| \begin{array}{c} x \\ \alpha_{ux} \\ \alpha_{vx} \\ \vdots \end{array} \begin{array}{c} y \\ \alpha_{uy} \\ \alpha_{vy} \\ \vdots \end{array} \dots \right\}$$

and the  $\alpha_{ux}$ 's are rational functions in variables  $t_u$ , one for each  $u \in T$ .

$$tm_{uv}^{uv}: \begin{array}{c} \omega \\ u \\ v \\ \vdots \end{array} \middle| \begin{array}{c} \dots \\ \alpha \\ \beta \\ \gamma \end{array} \mapsto \begin{array}{c} \omega \\ w \\ \vdots \end{array} \middle| \begin{array}{c} \dots \\ \alpha + \beta \\ \gamma \end{array}, \quad \begin{array}{c} \omega_1 | H_1 \\ T_1 | \alpha_1 \\ \omega_2 | H_2 \\ T_2 | \alpha_2 \end{array} \mapsto \begin{array}{c} \omega \\ z \\ \vdots \end{array} \middle| \begin{array}{c} \dots \\ \alpha + \beta + \langle \alpha \rangle \beta \\ \gamma \end{array}$$

$$hm_z^{xy}: \begin{array}{c} \omega \\ u \\ \vdots \end{array} \middle| \begin{array}{c} x \\ \alpha \\ \beta \\ \gamma \end{array} \mapsto \begin{array}{c} \omega \\ u \\ \vdots \end{array} \middle| \begin{array}{c} x \\ \alpha(1 + \langle \gamma \rangle / \epsilon) \\ \beta(1 + \langle \gamma \rangle / \epsilon) \end{array}$$

where  $\epsilon := 1 + \alpha$ ,  $\langle \alpha \rangle := \sum_v \alpha_v$ , and  $\langle \gamma \rangle := \sum_{v \neq u} \gamma_v$ , and let

$$R_{ux}^+ := \frac{1}{u} \middle| \begin{array}{c} x \\ t_u - 1 \end{array} \quad R_{ux}^- := \frac{1}{u} \middle| \begin{array}{c} x \\ t_u^{-1} - 1 \end{array}$$

On long knots,  $\omega$  is the Alexander polynomial!

**Why happy?** An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). *If there should be an Alexander invariant with a computable algebraic categorification, it is this one!*

See also  $\omega\epsilon\beta/regina$ ,  $\omega\epsilon\beta/caen$ ,  $\omega\epsilon\beta/newton$ .  
**May class:**  $\omega\epsilon\beta/aarhus$  **Class next year:**  $\omega\epsilon\beta/1350$   
**Paper:**  $\omega\epsilon\beta/kbh$



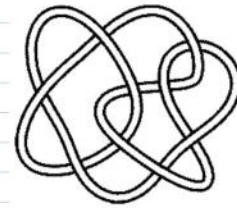


## The Most Important Missing Infrastructure Project in Knot Theory

January-23-12  
10:12 AM

An "infrastructure project" is hard (and sometimes non-glorious) work that's done now and pays off later.

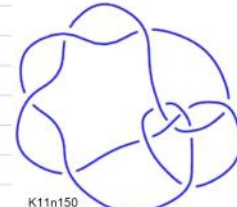
An example, and the most important one within knot theory, is the tabulation of knots up to 10 crossings. I think it precedes Rolfsen, yet the result is often called "the Rolfsen Table of Knots", as it is famously printed as an appendix to the famous book by Rolfsen. There is no doubt the production of the Rolfsen table was hard and non-glorious. Yet its impact was and is tremendous. Every new thought in knot theory is tested against the Rolfsen table, and it is hard to find a paper in knot theory that doesn't refer to the Rolfsen table in one way or another.



(KnotPlot image)

9\_42 is Alexander Stoimenow's favourite

A second example is the Hoste-Thistlethwaite tabulation of knots with up to 17 crossings. Perhaps more fun to do as the real hard work was delegated to a machine, yet hard it certainly was: a proof is in the fact that nobody so far had tried to replicate their work, not even to a smaller crossing number. Yet again, it is hard to overestimate the value of that project: in many ways the Rolfsen table is "not yet generic", and many phenomena that appear to be rare when looking at the Rolfsen table become the rule when the view is expanded. Likewise, other phenomena only appear for the first time when looking at higher crossing numbers.



K11n150

(Knotscape image)

But as I like to say, knots are the wrong object to study in knot theory. Let me quote (with some variation) my own (with Dancso) "WKO" paper:

Studying knots on their own is the parallel of studying cakes and pastries as they come out of the bakery - we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots (see [AKT-CFA]).



The interchange of I-95 and I-695, northeast of Baltimore. (more)

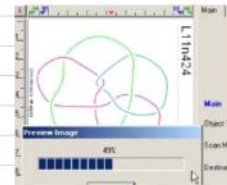
The right objects for study in knot theory are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids (which are already well-studied and tabulated) and even more so tangles and tangled graphs.

**Thus in my mind the most important missing infrastructure project in knot theory is the tabulation of tangles to as high a crossing number as practical.** This will enable a great amount of testing and experimentation for which the grounds are now still missing. The existence of such a tabulation will greatly impact the direction of knot theory, as many tangle theories and issues that are now ignored for the lack of scope, will suddenly become alive and relevant. The overall influence of such a tabulation, if done right, will be comparable to the influence of the Rolfsen table.



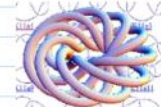
From [AKT-CFA]

Aside. What are tangles? Are they embedded in a disk? A ball? Do they have an "up side" and a "down side"? Are the strands oriented? Do we mod out by some symmetries or figure out the action of some symmetries? Shouldn't we also calculate the affect of various tangle operations (strand doubling and deletion, juxtapositions, etc.)? Shouldn't we also enumerate virtual tangles? w-tangles? Tangled graphs?



From [FastKh]

In my mind it would be better to leave these questions to the tabulator. Anything is better than nothing, yet good tabulators would try to tabulate the more general things from which the more special ones can be sieved relatively easily, and would see that their programs already contain all that would be easy to implement within their frameworks. Counting legs is easy and can be left to the end user. Determining symmetries is better done along with the enumeration itself, and so it should.



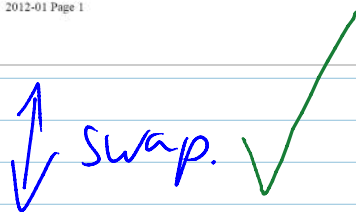
The Knot Atlas  
-Arjuna C. Eddy  
<http://katlas.org/>

**An even better tabulation** should come with a modern front-end - a set of programs for basic manipulations of tangles, and a web-based "tangle atlas" for an even easier access.

Overall this would be a major project, well worthy of your time.

(Source: <http://katlas.math.toronto.edu/drorbn/AcademicPensieve/2012-01/>)

2012-01 Page 1



Zurich

Pensieve header of the free-Lie meta-group-action structure for <http://www.math.toronto.edu/~drobn/TeX/NhaTRek-1305>

```
Get["http://www.math.toronto.edu/~drobn/TeX/NhaTRek-1305"];
Get["http://dromb.net/AcademicPensieve/2013-05/FreeLie.m"];
Get["http://dromb.net/AcademicPensieve/2013-05/muCalculus.m"];
```

```
u = {"u"}; v = {"v"}; BCH[u, v] @ (6)
LS[u + v, 2] == 1/2 u v + 1/12 u u v + 1/24 u u u v,
-1/720 u u u u v + 1/180 u u u v v + 1/120 u u u u v v + 1/360 u u u u u v + 1/720 u u v v v v,
```

```
-1/1440 u u u u v v + 1/360 u u u u v v + 1/240 u u u u u v + 1/720 u u u u u v + 1/1440 u u u u u v v
```

```
w = {"w"}; Print /@ (BCH[BCH[u, v], w], BCH[u, BCH[v, w]]):
```

```
LS[u + v + w, 2] == 1/2 (u v + u w + v w),
```

```
1/12 u u v + 1/12 u u w + 1/12 u v w + 1/12 u u v v + 1/12 u u w v + 1/12 u v w w,
```

```
LS[u + v + w, 2] == 1/2 (u v + u w + v w),
```

```
1/12 u u v + 1/12 u u w + 1/12 u v w + 1/12 u u v v + 1/12 u u w v + 1/12 u v w w,
```

```
Jv[BCH[u, v]] @ (4)
```

```
CMS[u, v, 2] == 1/2 u u v + 1/6 u u u v + 1/6 u u v v + 1/6 u u v v v + 1/6 u u v v v v
```

```
Testing hm[x,y,z] // tha[u,z] == tha[u,x] // tha[u,y] // hm[x,y,z]
```

```
Print /@ {
```

```
1 -> (t1 = M[x -> MakeLieSeries[u + b[u, v]], y -> MakeLieSeries[v + 2/3 b[u, v]]);
```

```
MakeCWSeries[CW["uu"] + CW["uvv"]];
```

```
2 -> (t2 = t1 // hm[x, y, z] // tha[u, z]);
```

```
3 -> (t3 = t1 // tha[u, x] // tha[u, y] // hm[x, y, z]);
```

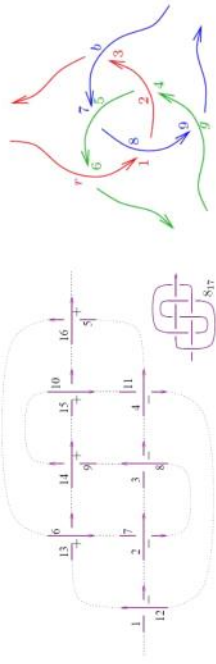
```
4 -> (t2 == t3);
```

```
1 -> M[x -> LS[u, uv, 0], y -> LS[v, 2/3 uv, 0], CMS[u, uv, uvv]]
```

```
2 -> M[z -> LS[u + v, 7/6 uv, 5/4 u u v - 13/12 u u v v], CMS[u, uv, 5/3 uv, 2/3 u u v]]
```

```
3 -> M[z -> LS[u + v, 7/6 uv, 5/4 u u v - 13/12 u u v v], CMS[u, uv, 5/3 uv, 2/3 u u v]]
```

```
4 -> True
```



### Demo 1 - The Knot 8<sub>17</sub>

```
mu1 = R'[12, 1] R'[8, 3] R'[4, 11] R'[16, 5] R'[6, 13] R'[14, 9] R'[10, 15]
M[{1 -> LS[-c, 0, 0], 2 -> LS[0, 0, 0], 3 -> LS[-8, 0, 0], 4 -> LS[0, 0, 0],
5 -> LS[g, 0, 0], 6 -> LS[0, 0, 0], 7 -> LS[-2, 0, 0], 8 -> LS[0, 0, 0], 9 -> LS[e, 0, 0],
10 -> LS[0, 0, 0], 11 -> LS[-4, 0, 0], 12 -> LS[0, 0, 0], 13 -> LS[6, 0, 0],
14 -> LS[0, 0, 0], 15 -> LS[a, 0, 0], 16 -> LS[0, 0, 0], CMS[0, 0, 0]]
```

```
Do[mu1 = mu1 // dm[1, k, 1], {k, 2, 16}]; mu1[W] @ (6)
```

```
CMS[0, -11, 0, -31/12, 0, -1351/360]
```

Compare with the Alexander polynomial:

$$\text{Series}\left[\text{Log}\left[-\frac{1}{x^3} + \frac{4}{x^2} + 11 - 8x + 4x^2 - x^3\right] / . x \rightarrow e^x, \{x, 0, 6\}\right]$$

$$-x^2 - \frac{31x^4}{12} - \frac{1351x^6}{360} + O[x]^7$$

### Demo 2 - The Borromean Tangle

```
mu2 = R'[r, 6] R'[2, 4] R'[g, 9] R'[5, 7] R'[b, 3] R'[8, 1];
```

```
(Do[mu2 = mu2 // dm[r, k, r], {k, 1, 3}]; Do[mu2 = mu2 // dm[g, k, g], {k, 4, 6}];
```

```
Do[mu2 = mu2 // dm[b, k, b], {k, 7, 9}]; mu2[r] @ (4); mu2[W] @ (4))
```

```
{LS[0, b, 2/3 b b g + b g r + 1/2 b g g],
```

```
1/6 b b b g + 1/2 b b g r + 1/2 b g g r + 1/4 b b g g + 1/2 b g r r + 1/6 b g g g],
```

```
CMS[0, 0, 2 b g r, b b g r - b g r + b g r + b g r - b g r]]
```