Cheat Sheet 3D Topology

Follows Hatcher's notes [Ha] and Hempel's book [He].

initiated 31/8/13; continues 2013-08; continued 2013-10; modified 2/10/13

Definition. M prime: $M = P\#Q \Rightarrow (P = S^3) \vee (Q = S^3)$. M Irreducible: an embedded 2-sphere in M bounds a 3-ball. (Irreducible $\Rightarrow$ Prime).

Theorem (Alexander, 1920s). S^3 is irreducible.

Theorem. Orientable, prime, not irreducible $\Rightarrow$ $S^2 \times S^1$. Nonorientable? Also $S^2 \tilde{\times} S^1$ (Klein 3D).

Theorem. Compact connected orientable 3-manifolds have unique decomposition into primes.

Proof. • Given a system of splitting spheres (sss) and a θ-partition of one member, at least one part will make an sss. • An sss can be simplified relative to a fixed triangulation τ: circle and single-edge-arc intersections with faces of τ can be eliminated. • The size of an sss is bounded by $4|\tau| + \operatorname{rank} H_1(M;\mathbb{Z}/2)$ and hence prime-decompositions exist. • Uniqueness. □

Nonorientable M? Same but $M\#(S^2 \times S^1) = M\#(S^2 \tilde{\times} S^1)$.

Theorem. If a covering is irreducible, so is the base. ([Ha] proof is fishy).

Examples. Lens spaces, surface bundles $F \to M \to S^1$ with $F \neq S^2, \mathbb{R}\mathrm{P}^2$. Yet $S^1 \times S^2/(x,y) \sim (\bar{x}, -y) = \mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$, a prime covers a sum.

Definition. $S \subset M^3$ a 2-sided surface, $S \neq S^2$, $S \neq D^2$. *Compressing disk for* S is a disk $D \subset M$ with $D \cap S = \partial D$. If for every compressing D there's a disk $D' \subset S$ with $\partial D' = \partial D$, S is *incompressible.*

Claims. • $\pi_1(S) \hookrightarrow \pi_1(M) \Rightarrow S$ incompressible. • No incompressibles in $\mathbb{R}^3/S^3$. • In irreducible M^3, T^2 is 2-sided incompressible iff T bounds a $D^2 \times S^1$ or T is contained in a B^3. • A T^2 in S^3 bounds a $D^2 \times S^1$ on at least one side. • $S \subset M$ incompressible $\Rightarrow$ (M irreducible iff $M|S$ irreducible). • S a collection of disjoint incompressibles or disks or spheres in M, $T \subset M|S$. Then T is incompressible in M iff in $M|S$.

Dehn's Lemma (Dehn 1910 (wrong), Papakyriakopoulos 1950s). M a 3-manifold, $f\colon B^2 \to M$ s.t. for some neighborhood A of ∂B^2 in B^2 the restriction $F|_A$ is an embedding and $f^{-1}(f(A)) = A$. Then $f|_{\partial B^2}$ extends to an embedding $g\colon B^2 \to M$.

The Loop Theorem (Stallings 1960, implies Dehn's lemma). M a 3-manifold, F a connected 2-manifold in ∂M, $\ker(\pi_1(F) \to \pi_1(M) \not\subset N \triangleleft \pi_1(F)$. Then there is a proper embedding $g\colon (B^2, \partial B^2) \to (M, F)$ s.t. $[g|_{\partial B^2}] \notin N$.

The Sphere Theorem. M orientable 3-manifold, N a $\pi_1(M)$-invariant proper subgroup of $\pi_2(M)$. Then there is an embedding $g\colon S^2 \to M$ s.t. $[g] \notin N$.

To do. • JSJ.