Pensieve header: A concise implementation of the FastKh algorithm; continues pensieve://2013-06/; annotated version in pensieve://2017-08/.

Full sources at http://drorbn.net/AcademicPensieve/2013-07/.

```
<< KnotTheory`
```

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
SetAttributes[{P, S}, Orderless]; dot /: dot[_]^k_ /; k ≥ 2 := 0;
(σ_S)[i_] := σ[i] = First@Cases[σ, P[i, j_] :→ j];
```

```
EC[λ_List] := Module[{ρ, ec = λ}, (* "Finding Equivalence Classes" *)
   Do[ ρ = First /@ Position[ec, i];
    ec = Append[Delete[ec, List /@ ρ], Union @@ (ec[[ρ]])],
    {i, Union @@ λ} ]; ec];
EC[λ__S] := EC[Join[λ] /. S | P → List];
ECP[λ__] := Union @@ Replace[EC[λ], c_ :→ ((# → First[c]) & /@ c), {1}] ;
```

```
VC[β_S, μ_S, τ_S] := VC[β, μ, τ] = Module[{ins, outs, p, χs, h, dec, dots, law} ,
    ins = First /@ Join[EC[β, μ], EC[μ, τ]];
    outs = First /@ EC[β, τ]; p = ECP[β, μ, τ];
    χs = Times @@ (h /@ Join[ins, outs] /. p);
    χs *= PowerExpand[(Times @@ (h /@ (Last /@ p)))^(-1/2)];
    dec = χs /. h[i_]^χ_. :→ (2 dot[i])^((2-χ)/2);
    dec *= Product[If[i == (i /. p), 1, dot[i] + dot[i /. p]], {i, outs}];
    Expand[dots * # /. law] & /.
     {dots → Expand[dec], law → Table[dot[i] → dot[i /. p], {i, Union[ins]}]} ];
```

```
m0[i_, j_][σ_S] := m0[i, j][σ] = If[σ[i] == j, DeleteCases[σ, P[i, j]],
    Append[DeleteCases[σ, P[i, _] | P[_, j]], P[σ[i], σ[j]]] ];
m[i_, j_][σ_S] := m0[i, j][σ] * If[σ[i] == j, {q, q^-1}, {1}];
m[i_, j_][q^k_. σ_S] := q^k m[i, j][σ];
```

```
m[i_, j_][Cob[β_S, τ_S, dots_]] := Module[{p, ijdot, np, ndots, x},
   p = ECP[β, τ]; ijdot = dot@Min[i, j]; np = ECP[m0[i, j][β], m0[i, j][τ]];
   ndots = Which[β[i] == j ∧ τ[i] == j, ( ijdot    0   ),
                                        (   1    ijdot ),
     β[i] == j ∧ τ[i] ≠ j, {{1, ijdot}},
     β[i] ≠ j ∧ τ[i] == j, {{ijdot}, {1}}, β[i] ≠ j ∧ τ[i] ≠ j,
     {{If[(i /. p) ≠ (j /. p), 1, dot[β[i]] + dot[τ[i]]]}} ];
   ndots = Expand[dots * ndots] /. dot[k_] :→
      dot[k /. {i → β[i], j → β[j]} /. {i → τ[i], j → τ[j]} /. np ];
   If[β[i] == j ∧ τ[i] == j, Coefficient[ndots /. ijdot → x, x], ndots]];
```

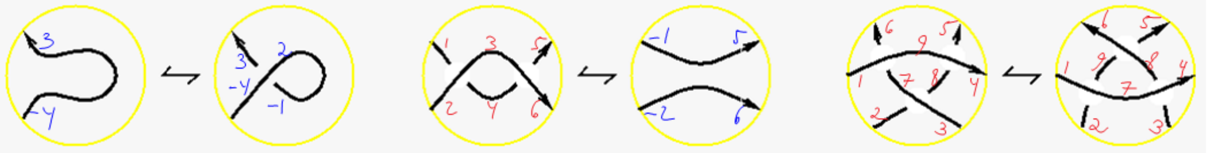
```
m[i_, j_][Kom[Ω_, d_]] := Kom[
   Flatten /@ Map[m[i, j], Ω, {2}],
   Table[If[Length@Ω⟦k⟧ == 0 ∨ Length@Ω⟦k + 1⟧ == 0, 0,
     Table[m[i, j][Cob[Ω⟦k, b⟧, Ω⟦k + 1, a⟧, d⟦k, a, b⟧]] /. q → 1],
       {a, Length@Ω⟦k + 1⟧}, {b, Length@Ω⟦k⟧}
      ] // ArrayFlatten ],
    {k, Length@d}] ];
```

```
(Kom[Ω_, d_] // Cob[q^(p1_.) β_, q^(p2_.) τ_, 1]) := Module[{L, ρ, δ, k},
  L = Length[Ω]; ρ[k_] := ρ[k] = Length[Ω⟦k⟧]; ρ[0] = ρ[L + 1] = 0;
  Kom[
   MapThread[Join, List @@@ {
      Append[Ω /. σ_S :→ q^p1 Join[β, σ], {}],
      Prepend[Ω /. σ_S :→ q^p2 Join[τ, σ], {}] }],
   Table[
    If[ρ[k] + ρ[k - 1] == 0 ∨ ρ[k + 1] + ρ[k] == 0, 0,
     δ = Table[0, {ρ[k + 1] + ρ[k]}, {ρ[k] + ρ[k - 1]}];
     If[ρ[k] ρ[k + 1] ≠ 0, δ⟦1 ;; ρ[k + 1], 1 ;; ρ[k]⟧ = d⟦k⟧];
     If[ρ[k] ≠ 0,
      δ⟦ρ[k + 1] + 1 ;; ρ[k + 1] + ρ[k], 1 ;; ρ[k]⟧ = (-1)^k IdentityMatrix[ρ[k]]];
     If[ρ[k - 1] ρ[k] ≠ 0, δ⟦ρ[k + 1] + 1 ;; ρ[k + 1] + ρ[k],
        ρ[k] + 1 ;; ρ[k] + ρ[k - 1]⟧ = d⟦k - 1⟧];
     δ
    ], {k, L} ] ] ]
```

```
Contract[kom_Kom] := Module[{Ω, d, L, ρ, k, done, a, b, ϕ, γδ},
   {Ω, d} = List @@ kom; L = Length@d; ρ[k_] := Length@Ω⟦k⟧;
   For[k = 1, k ≤ L, ++k,
    done = False; While[! done, done = True;
     For[a = 1, a ≤ ρ[k + 1], ++a, For[b = 1, b ≤ ρ[k], ++b,
       If[NumberQ[ϕ = d⟦k, a, b⟧] ∧ ϕ ≠ 0 ∧ Ω⟦k + 1, a⟧ == Ω⟦k, b⟧,
        done = False;
        If[ρ[k] ≤ 1 ∨ ρ[k + 1] ≤ 1, d⟦k⟧ = 0,
         γδ = Table[VC[Ω⟦k, n⟧ /. q → 1, Ω⟦k + 1, a⟧ /. q → 1, Ω⟦k + 1, m⟧ /. q → 1][
             d⟦k, a, n⟧ d⟦k, m, b⟧], {m, ρ[k + 1]}, {n, ρ[k]} ];
         d⟦k⟧ = Expand@Drop[d⟦k⟧ - ϕ^-1 γδ, {a}, {b}]];
        Ω⟦k⟧ = Drop[Ω⟦k⟧, {b}]; Ω⟦k + 1⟧ = Drop[Ω⟦k + 1⟧, {a}];
        If[k > 1 ∧ ρ[k - 1] > 0, d⟦k - 1⟧ = Drop[d⟦k - 1⟧, {b}]];
        If[k < L ∧ ρ[k + 2] > 0, d⟦k + 1⟧ = Drop[d⟦k + 1⟧, {}, {a}]];
        If[a ≤ ρ[k + 1], --a]; b = ρ[k] ] ]] ] ];
   Kom[Ω, d] ];
```

```
Kom[] = Kom[{{S[]}}, {}];
Cob@Xp[i_, j_, k_, l_] := Cob[q S[P[-i, j], P[k, -l]], q^2 S[P[-i, -l], P[j, k]], 1];
Cob@Xm[i_, j_, k_, l_] := Cob[q^-2 S[P[-i, -j], P[k, l]], q^-1 S[P[-i, l], P[-j, k]], 1];
Cob[x_X] := Cob[If[PositiveQ[x], Xp @@ x, Xm @@ x]];
```

```
KhComplex[L_] := Module[
   {pd = PD[L], kom = Kom[], inside = {}, pos},
   While[Length[pd] > 0,
    pos = Last[Ordering[(Length[(List @@ #) ⋂ inside]) & /@ pd]];
    kom = kom // Cob[pd[[pos]]];
    (kom = Contract[kom // m[#, -#]]) & /@ ((List @@ pd[[pos]]) ⋂ inside);
    inside = inside ⋃ (List @@ pd[[pos]]); pd = Drop[pd, {pos}] ];
   kom ];
KhPoly[L_] := Expand[t^(-Length@Select[PD@L, NegativeQ]+Range[0,Crossings[L]]).
    (List @@ Plus @@@ First @ KhComplex[L]) /. S[] → 1]
```



```
Kom[] // Cob[q S[P[-1, 2], P[3, -4]], q^2 S[P[-1, -4], P[2, 3]], 1] // m[-1, 2] // Contract
```

```
Kom[{{S[P[-4, 3]]}, {}}, {0}]
```

```
Kom[] // Cob[Xm[1, 2, 4, 3]] // Cob[Xp[4, 6, 5, 3]] // m[3, -3] // m[4, -4] // Contract
```

```
Kom[{{}, {S[P[-2, 6], P[-1, 5]]}, {}}, {0, 0}]
```

```
R31 =
 Kom[] // Cob[Xp[7, 9, 6, 1]] // Cob[Xp[8, 4, 5, 9]] // Cob[Xm[2, 3, 8, 7]] // m[-7, 7] //
    m[-8, 8] // m[-9, 9] // Contract
```

```
Kom[{{}, {q S[P[-3, -2], P[-1, 4], P[5, 6]], q S[P[-3, 4], P[-2, 5], P[-1, 6]]},
  {q^2 S[P[-3, 4], P[-2, -1], P[5, 6]], q^2 S[P[-3, -2], P[-1, 6], P[4, 5]]},
  {q^3 S[P[-3, 6], P[-2, -1], P[4, 5]]}}, {0, {{1, -1}, {1, -1}}, {{1, -1}}}]
```

```
R32 =
 Kom[] // Cob[Xp[2, 7, 9, 1]] // Cob[Xp[3, 4, 8, 7]] // Cob[Xm[9, 8, 5, 6]] // m[-7, 7] //
    m[-8, 8] // m[-9, 9] // Contract
```

```
Kom[{{}, {q S[P[-3, -2], P[-1, 4], P[5, 6]], q S[P[-3, 4], P[-2, 5], P[-1, 6]]},
  {q^2 S[P[-3, 4], P[-2, -1], P[5, 6]], q^2 S[P[-3, -2], P[-1, 6], P[4, 5]]},
  {q^3 S[P[-3, 6], P[-2, -1], P[4, 5]]}}, {0, {{1, -1}, {1, -1}}, {{1, -1}}}]
```

```
R31 == R32
```

```
True
```

```
K = TorusKnot[9, 5]; {TubePlot[K, ImageSize → 80] // Rasterize, KhPoly[K]} // Timing
```

{933.556784,

{ ,

$$q^{31} + q^{33} + q^{35} t^{2} + q^{39} t^{3} + q^{37} t^{4} + q^{39} t^{4} + q^{41} t^{5} + q^{43} t^{5} + q^{39} t^{6} + q^{41} t^{6} + q^{43} t^{7} + q^{45} t^{7} + q^{41} t^{8} + 2 q^{43} t^{8} + q^{45} t^{9} + 2 q^{47} t^{9} + 2 q^{45} t^{10} + 3 q^{49} t^{11} + 2 q^{47} t^{12} + 2 q^{49} t^{12} + q^{53} t^{12} + 3 q^{51} t^{13} + 2 q^{53} t^{13} + q^{49} t^{14} + 2 q^{51} t^{14} + q^{55} t^{14} + 2 q^{53} t^{15} + 3 q^{55} t^{15} + 2 q^{53} t^{16} + q^{57} t^{16} + q^{59} t^{16} + 3 q^{57} t^{17} + q^{55} t^{18} + q^{57} t^{18} + q^{61} t^{18} + 2 q^{59} t^{19} + q^{61} t^{19} + q^{59} t^{20} + q^{63} t^{20} + q^{63} t^{21}$$

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