Marcos Marino on Integrality properties of link invariants from topological-strings

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8:32 AM

Basic idea: Motivated by

Gopakumar-Vafa & Ooguri-Vafa:

(knot invariants) $\Longleftrightarrow$ GW-type invariants.

$\Updownarrow$

integrality props of knot invariants $\Longleftarrow$ integrality properties GV invariants BPS invariants

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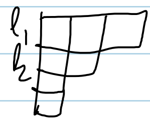
Group theoretical ingredients

R: irrep of S_l (symmetric group on l elements)

$\Longleftrightarrow$ Young diagram w/ l boxes

$\Longleftrightarrow$ $\{l_i\}_{i=1 \ldots r(R)}$



$$l(R) = \sum_{i=1}^{r(R)} l_i \qquad K_R = \sum_{i=1}^{r(R)} l_i(l_i - 2i + 1)$$

Generating functions for coloured objects:

SYMMETRIC POLYNOMIALS in infinite number of variables $\{V_i\}_{i \geq 1}$

basis: Schur Polynomials $S_R(V)$

$$V = \begin{pmatrix} V_1 & & & \\ & V_2 & & \\ & & \ddots & \\ & & & V \end{pmatrix} \qquad S_R(V) = \mathrm{tr}_R V$$

Examples $S_{\square}(V) = \sum v_i$

$$S_{\square\square}(V) = \tfrac{1}{2}(\mathrm{tr}V)^2 + \tfrac{1}{2}\mathrm{tr}V^2 = \sum_{i \geq 1} \tfrac{1}{2} v_i^2 + \sum_{i,j \geq 1} \tfrac{1}{2} v_i v_j$$

$$S_{\boxminus}(V) = \tfrac{1}{2}(\mathrm{tr}V)^2 - \tfrac{1}{2}\mathrm{tr}V^2$$

Composite representations (R_1, R_2) pairs of Young tableaux think using $U(N)$:

$$S_{R_1}(V) S_{R_2}(V) = \sum N^{R}_{R_1 R_2} S_R(V)$$

Littlewood-Richardson coeffs,

also $R_1 \otimes R_2 = \sum N^{R}_{R_1 R_2} \cdot R$

$$(R, S) = \sum_{uvw} (-1)^{\ell(u)} N^{R}_{uv} N^{S}_{u^T w} V \otimes \overline{W}$$

complex conjugate

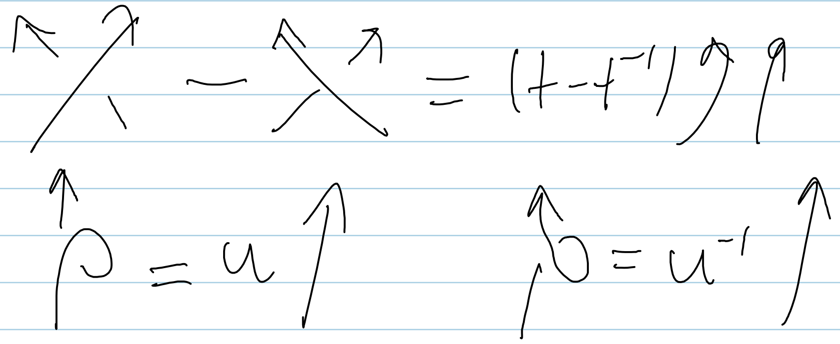
flip rows & columns

$= R \otimes S +$ lower (fewer box) corrections

Colored HOMFLY Consider an oriented link L with projection D_L.

The skein of the plane is link diagrams mod skein relations: (variables t & u)

skein relations: (with vars t & u)



under this, $D_\ell = \langle D_L \rangle \bigcirc$

$$P_L(t,u) = u^{-\bar{w}(D_L)} \langle D_L \rangle$$

$\bar{w}(D_L)$ = self-writhe = sum of signs over xings of a component w/ itself.

$\Longrightarrow$ differs from the standard def of HOMFLY by a factor $u^{2(\text{total linking number})}$

The HOMFLY invariant:

$$\mathcal{H}(L) = P_L(u,t)\,\mathcal{H}(\bigcirc)$$

with $\mathcal{H}(\bigcirc) = \dfrac{v-v^{-1}}{t-t^{-1}}$

Higher reps: (use satellites)

K = framed knot = a knotted annulus 

P = knot drawn in the annulus 

$K * P$: the P-satellite of K.

Skein of the annulus: there is a basis labeled by composite reps (R,S)

$$\Longrightarrow P_{(R,S)}$$

Define

$$\mathcal{H}_{(R,S)}(K) = \mathcal{H}(K * P_{(R,S)})$$

can be generalized to n components

<u>Examples</u>

$$P_{\square} = \uparrow$$

$$P_{\square\square} = \frac{1}{1+t^2}\left(\uparrow\uparrow + t\right.$$  $$\left.\right)$$

$$P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{1+t^2}\left(\uparrow\uparrow - t^{-1}\right.$$  $$\left.\right)$$

$$P_{(\square,\square)} = \uparrow\downarrow - 1$$

<u>properties</u>

1. Orientation: $P_{(R,S)} \longrightarrow P_{(S,R)}$

$$\mathcal{H}_{(R,S)}(K) = \mathcal{H}_{(S,R)}(\overline{K})$$

similarly for reversing an individual component.

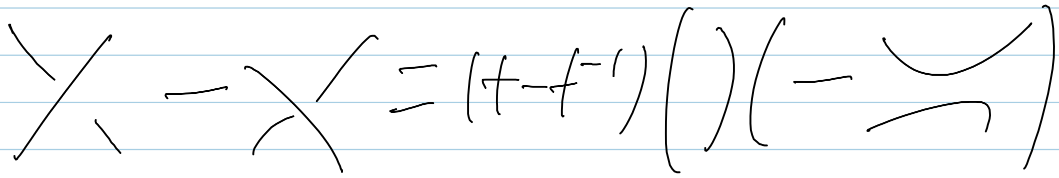
2. This is the same as colored-Homfly as it comes from quantum groups, under

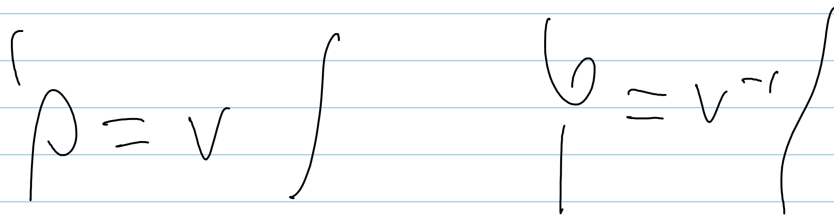
Some change of variables:

$$t = q^{1/2} \qquad v = q^{N/2}$$

$$\mathcal{H}_R(\bigcirc) = \dim_q(R)$$

Kauffman Invariant Unoriented diags in the plane, modulo





Let L be an unoriented link w/ plane diagram E_L, set $E_L = \langle E_L \rangle \bigcirc$

$$F_L(t, v) = v^{-w(E_L)} \langle E_L \rangle$$

"the Kauffman poly"

$$g(L) = F_L \cdot g(O) = 1 + \frac{v - v^{-1}}{t - t^{-1}}$$

There is a similar procedure for the coloured Kauffman ~ Skein of annulus has a basis also labeled by Young tableaux (Beliakova – Blanchet)

$$g_R(O) = q\dim R \text{ in } U_q(so(N))$$

$$t = q^{1/2} \qquad v = t^{N-1}$$

<u>Examples</u>

3_1

$$z = t - t^{-1}$$

$$P_{3_1}(t,v) = 2v^2 - v^4 - z^2v^2$$

$$F_{3_1}(t,v) = 2v^2 - v^4 + z(-v^3 + v^5) + z^2(v^2 - v^4)$$