Cheat Sheet β

The original β-calculus: With $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$,

$$\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array} * \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array} \underset{\beta}{=} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & A_1 & 0\\ T_2 & 0 & A_2\end{array} \qquad \begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \gamma\end{array} \xrightarrow[\beta]{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \gamma\end{array}\right) /\!\!/ \left(\substack{u,v\\ \to w}\right) \qquad \rho^\pm_{ux} \underset{\beta}{=} \begin{array}{c|c}1 & x\\ \hline u & t_u^{\pm1}-1\end{array}$$

$$\begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \gamma\end{array} \xrightarrow[\beta]{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\beta+\langle\alpha\rangle\beta & \gamma\end{array} \qquad \begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \beta\\ T & \gamma & \delta\end{array} \xrightarrow[\beta]{sw^{ux}_{th}} \begin{array}{c|cc}\omega\epsilon & x & H\\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon)\\ T & \gamma/\epsilon & \delta-\gamma\beta/\epsilon\end{array}$$

Constraints. • Column sums are monomials minus 1. • At $t_*=1$, $\omega=1$ and $A=0$.

β-better calculus: Constraints. • Sum of column x is $(\sigma_x-1)w$. • $\omega^{k-1}\mid\Lambda^kA$. • At $t_*=1$, $\omega=1$ and $A=0$.

$$\begin{array}{c|c}\omega_1 & H_1\\ - & \sigma_1\\ \hline T_1 & A_1\end{array} * \begin{array}{c|c}\omega_2 & H_2\\ - & \sigma_2\\ \hline T_2 & A_2\end{array} \underset{\beta_b}{=} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ - & \sigma_1 & \sigma_2\\ \hline T_1 & \omega_2A_1 & 0\\ T_2 & 0 & \omega_1A_2\end{array} \qquad \begin{array}{c|c}\omega & H\\ - & \sigma\\ \hline u & \alpha\\ v & \beta\\ T & \gamma\end{array} \xrightarrow[\beta_b]{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ - & \sigma\\ \hline w & \alpha+\beta\\ T & \gamma\end{array}\right) /\!\!/ \left(\substack{u,v\\ \to w}\right) \qquad \rho^\pm_{ux} \underset{\beta_b}{=} \begin{array}{c|c}1 & x\\ - & t_u^{\pm1}\\ \hline u & t_u^{\pm1}-1\end{array}$$

$$\begin{array}{c|ccc}\omega & x & y & H\\ - & \sigma_x & \sigma_y & \sigma\\ \hline T & \alpha & \beta & \gamma\end{array} \xrightarrow[\beta_b]{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ - & \sigma_x\sigma_y & \sigma\\ \hline T & \alpha+\sigma_x\beta & \gamma\end{array} \qquad \begin{array}{c|cc}\omega & x & H\\ - & \sigma_x & \sigma\\ \hline u & \alpha & \beta\\ T & \gamma & \delta\end{array} \xrightarrow[\beta_b]{sw^{ux}_{th}} \begin{array}{c|cc}\omega+\alpha & x & H\\ - & \sigma_x & \sigma\\ \hline u & \sigma_x\alpha & \sigma_x\beta\\ T & \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega}\end{array} =: \begin{array}{c|c}\cdot & -\\ \cdot & -\\ \hline \vert & \begin{pmatrix}\sigma_x & 0\\ 0 & 1\end{pmatrix}\cdot A^{ux}\end{array}$$

Note. $A^{ux} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta+\frac{\alpha\delta-\gamma\beta}{\omega}\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \frac{(\omega+\alpha)\delta-\gamma\beta}{\omega}\end{pmatrix} = \frac{1}{\omega}\left[(\omega+\alpha)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} - \begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\begin{pmatrix}\alpha & \beta\end{pmatrix}\right] = \frac{1}{\omega}\left[(\omega+a_{ux})A - a_{*x}a_{u*}\right].$

Claim. $\omega^{k-1}\mid\Lambda^kA$ and $\omega^k\mid\Lambda^{k+1}A$ implies $(\omega+\alpha)^{k-1}\mid\Lambda^kA^{ux}$, with $\alpha=a_{ux}$.

Proof. With $\bar u\in T^k$ and $\bar x\in H^k$, ω^k divides $\begin{vmatrix}\omega & 0\\ 0 & a_{\bar u\bar x}\end{vmatrix}$ and $\begin{vmatrix}a_{ux} & a_{u\bar x}\\ a_{\bar ux} & a_{\bar u\bar x}\end{vmatrix}$ and hence their sum, $\begin{vmatrix}\omega+\alpha & a_{u\bar x}\\ a_{\bar ux} & a_{\bar u\bar x}\end{vmatrix} = (\omega+\alpha)\begin{vmatrix}1 & 0\\ 0 & a_{\bar u\bar x}-\frac{1}{\omega+\alpha}a_{\bar ux}a_{u\bar x}\end{vmatrix} = \frac{1}{(\omega+\alpha)^{k-1}}\left|(\omega+\alpha)a_{\bar u\bar x}-a_{\bar ux}a_{u\bar x}\right|$. So $\frac{1}{(\omega+\alpha)^{k-1}}\left|\frac{1}{\omega}\left[(\omega+\alpha)a_{\bar u\bar x}-a_{\bar ux}a_{u\bar x}\right]\right|$ is integral. $\square$

That is, with $A_{\bar u;\bar x}$ denoting minors, if $\omega^{k-1}\mu_{\bar u;\bar x}=A_{\bar u;\bar x}$ and $\omega^k\mu_{u\bar u;x\bar x}=A_{u\bar u;x\bar x}$, then $(\omega+\alpha)^{k-1}(\mu_{\bar u;\bar x}+\mu_{u\bar u;x\bar x})=A^{ux}_{\bar u;\bar x}$.

Relations. • $\rho^+_{ux}\rho^-_{vy} /\!\!/ tm^{uv}_w /\!\!/ hm^{xy}_z = t\epsilon_w h\epsilon_z$. • $\rho^{s_1}_{ux}\rho^{s_2}_{vy}\rho^{s_2}_{wz} /\!\!/ tm^{vw}_v /\!\!/ hm^{xy}_x /\!\!/ sw^{ux}_{th} = \rho^{s_2}_{vx}\rho^{s_2}_{wz}\rho^{s_1}_{uy} /\!\!/ tm^{vw}_v /\!\!/ hm^{xy}_x$.

Λ-calculus: $\Lambda(T;H)=R(T)\otimes(\Lambda(T)\otimes\Lambda(H))_=\times\Sigma(T)^H$, with $R(T)$ Laurent polynomials in $\{t_u\}_{u\in T}$ and $\Sigma(T)$ its units. Generic element is $L=(\lambda,\sigma=(x\to\sigma_x))$.

$tm^{uv}_w:\ u,v\to w,\ t_u,t_v\to t_w$ $\qquad hm^{xy}_z:\ x\to z,\ y\to\sigma_xz,\ \sigma_z:=\sigma_x\sigma_y$ $\qquad sw^{ux}_{th}: L\mapsto((1+i_u\otimes i_x)\lambda /\!\!/ (u\to\sigma_xu),\sigma)$

$L_1\cdot L_2=(\lambda_1(\wedge\otimes\wedge)\lambda_2,\sigma_1\cup\sigma_2)$ $\qquad \rho^\pm_{ux}=(1+(t_u^{\pm1}-1)ux,(x\to t_u^{\pm1}))$

To do. • Consider a verification program. • Add dm formulas. • Add Burau calculus. • Add the MVA formula