Pensieve header: Demo of the free-Lie meta-group-action structure for http://www.math.toronto.edu/~drorbn/Talks/NhaTrang-1305/.

```
Get["http://drorbn.net/AcademicPensieve/2013-05/FreeLie.m"];
Get["http://drorbn.net/AcademicPensieve/2013-05/muCalculus.m"];
```

```
u = ⟨"u"⟩; v = ⟨"v"⟩; BCH[u, v]@{6}
```

$$\mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{\overline{uv}}{2},\ \frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{\overline{uv}v},\ \frac{1}{24}\overline{u\overline{\overline{uv}v}},$$
$$-\frac{1}{720}\overline{u\,u\,\overline{u\overline{uv}}}+\frac{1}{180}\overline{u\,u\,\overline{\overline{uv}v}}+\frac{1}{180}\overline{u\,\overline{\overline{uv}v}\,v}+\frac{1}{120}\overline{\overline{uv}\,\overline{uv}v}+\frac{1}{360}\overline{u\overline{uv}\,\overline{uv}}-\frac{1}{720}\overline{\overline{\overline{uv}v}\,vv},$$
$$-\frac{\overline{u\,u\,u\,\overline{\overline{uv}v}}}{1440}+\frac{1}{360}\overline{u\,u\,\overline{\overline{\overline{uv}v}\,v}}+\frac{1}{240}\overline{u\overline{uv}\,\overline{\overline{uv}v}}+\frac{1}{720}\overline{u\,u\overline{uv}\,\overline{uv}}-\frac{\overline{u\,\overline{\overline{uv}v}\,vv}}{1440}\Big]$$

```
w = ⟨"w"⟩; Print /@ {BCH[BCH[u, v], w], BCH[u, BCH[v, w]]};
```

$$\mathrm{LS}\Big[\overline{u}+\overline{v}+\overline{w},\ \frac{\overline{uv}}{2}+\frac{\overline{uw}}{2}+\frac{\overline{vw}}{2},\ \frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{u\overline{uw}}+\frac{1}{3}\overline{u\overline{vw}}+\frac{1}{12}\overline{v\overline{vw}}+\frac{1}{12}\overline{\overline{uv}v}+\frac{1}{6}\overline{\overline{uw}v}+\frac{1}{12}\overline{\overline{uw}w}+\frac{1}{12}\overline{\overline{vw}w}\Big]$$
$$\mathrm{LS}\Big[\overline{u}+\overline{v}+\overline{w},\ \frac{\overline{uv}}{2}+\frac{\overline{uw}}{2}+\frac{\overline{vw}}{2},\ \frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{u\overline{uw}}+\frac{1}{3}\overline{u\overline{vw}}+\frac{1}{12}\overline{v\overline{vw}}+\frac{1}{12}\overline{\overline{uv}v}+\frac{1}{6}\overline{\overline{uw}v}+\frac{1}{12}\overline{\overline{uw}w}+\frac{1}{12}\overline{\overline{vw}w}\Big]$$

```
Jv[BCH[u, v]]@{4}
```

$$\mathrm{CWS}\Big[\widehat{v},\ \widehat{uv},\ \frac{\widehat{uuv}}{2}-\frac{\widehat{uvv}}{2},\ \frac{\widehat{uuuv}}{6}-\frac{\widehat{uuvv}}{4}-\frac{\widehat{uvuv}}{2}+\frac{\widehat{uvvv}}{6}\Big]$$

Testing hm[x,y,z] // tha[u,z] ≡ tha[u,x] // tha[u,y] // hm[x,y,z]

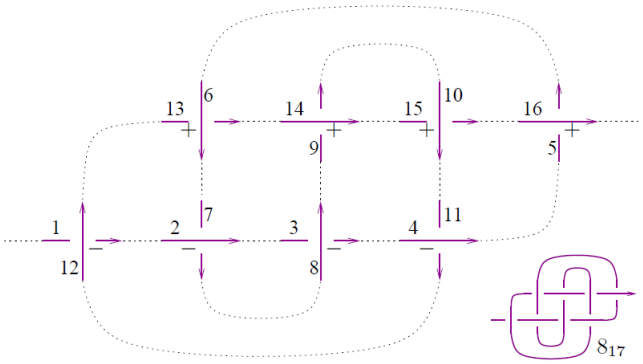
```
Print /@ {
   1 → (t1 = M[{x → MakeLieSeries[u + b[u, v]], y → MakeLieSeries[v + 2/3 b[u, v]]},
       MakeCWSeries[CW["uu"] + CW["uvv"]]]),
   2 → (t2 = t1 // hm[x, y, z] // tha[u, z]),
   3 → (t3 = t1 // tha[u, x] // tha[u, y] // hm[x, y, z]),
   4 → (t2 ≡ t3)};
```

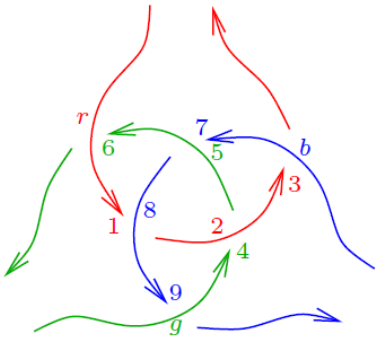
$$1 \to \mathrm{M}\Big[\{x \to \mathrm{LS}\Big[\overline{u},\ \overline{uv},\ 0\Big],\ y \to \mathrm{LS}\Big[\overline{v},\ \frac{2\,\overline{uv}}{3},\ 0\Big]\},\ \mathrm{CWS}\Big[0,\ \widehat{uu},\ \widehat{uvv}\Big]\Big]$$

$$2 \to \mathrm{M}\Big[\{z \to \mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{7\,\overline{uv}}{6},\ -\frac{5}{4}\overline{u\overline{uv}}-\frac{13}{12}\overline{\overline{uv}v}\Big]\},\ \mathrm{CWS}\Big[\widehat{u},\ \widehat{uu}-\frac{5\,\widehat{uv}}{3},\ \frac{\widehat{uuv}}{2}+\frac{2\,\widehat{uvv}}{3}\Big]\Big]$$

$$3 \to \mathrm{M}\Big[\{z \to \mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{7\,\overline{uv}}{6},\ -\frac{5}{4}\overline{u\overline{uv}}-\frac{13}{12}\overline{\overline{uv}v}\Big]\},\ \mathrm{CWS}\Big[\widehat{u},\ \widehat{uu}-\frac{5\,\widehat{uv}}{3},\ \frac{\widehat{uuv}}{2}+\frac{2\,\widehat{uvv}}{3}\Big]\Big]$$

$$4 \to \mathrm{True}$$





Demo 1 - The Knot 8_{17}

```
μ1 = R⁻[12, 1] R⁻[2, 7] R⁻[8, 3] R⁻[4, 11] R⁺[16, 5] R⁺[6, 13] R⁺[14, 9] R⁺[10, 15]
```

```
M[{1 → LS[-c̄, 0, 0], 2 → LS[0, 0, 0], 3 → LS[-8̄, 0, 0], 4 → LS[0, 0, 0], 5 → LS[ḡ, 0, 0], 6 → LS[0, 0, 0],
  7 → LS[-2̄, 0, 0], 8 → LS[0, 0, 0], 9 → LS[ē, 0, 0], 10 → LS[0, 0, 0], 11 → LS[-4̄, 0, 0],
  12 → LS[0, 0, 0], 13 → LS[6̄, 0, 0], 14 → LS[0, 0, 0], 15 → LS[ā, 0, 0], 16 → LS[0, 0, 0]}, CWS[0, 0, 0]]
```

```
Do[μ1 = μ1 // dm[1, k, 1], {k, 2, 16}]; μ1[W]@{6}
```

$$\mathrm{CWS}\left[0, -\overline{11}, 0, -\frac{31\,\overline{1111}}{12}, 0, -\frac{1351\,\overline{111111}}{360}\right]$$

Compare with the Alexander polynomial:

```
Series[Log[-1/X^3 + 4/X^2 - 8/X + 11 - 8 X + 4 X^2 - X^3 /. X → e^x], {x, 0, 6}]
```

$$-x^2 - \frac{31\,x^4}{12} - \frac{1351\,x^6}{360} + O[x]^7$$

Demo 2 - The Borromean Tangle

```
μ2 = R⁻[r, 6] R⁺[2, 4] R⁻[g, 9] R⁺[5, 7] R⁻[b, 3] R⁺[8, 1];
(Do[μ2 = μ2 // dm[r, k, r], {k, 1, 3}]; Do[μ2 = μ2 // dm[g, k, g], {k, 4, 6}];
 Do[μ2 = μ2 // dm[b, k, b], {k, 7, 9}]; {μ2[r]@{4}, μ2[W]@{4}})
```

$$\left\{\mathrm{LS}\left[0, \overline{bg}, \frac{1}{2}\,\overline{b\,\overline{bg}} + \overline{b\,\overline{gr}} + \frac{1}{2}\,\overline{bgg}, \frac{1}{6}\,\overline{b\,\overline{b\,\overline{bg}}} + \frac{1}{2}\,\overline{b\,\overline{b\,\overline{gr}}} + \frac{1}{2}\,\overline{b\,\overline{g\,\overline{gr}}} + \frac{1}{4}\,\overline{b\,\overline{bgg}} + \frac{1}{2}\,\overline{b\,\overline{grr}} + \frac{1}{6}\,\overline{\overline{bggg}}\right],\right.$$
$$\left.\mathrm{CWS}\left[0, 0, 2\,\overline{bgr}, \overline{bbgr} - \overline{bgbr} + \overline{bggr} - \overline{bgrg} + \overline{bgrr} - \overline{brgr}\right]\right\}$$