

Free Lie Algebras Tests

Pensieve header: Tests for FreeLie.nb.

Global Definitions

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2013-05"];
<< FreeLie.m
```

Words and Lyndon Words

```
{LyndonQ[AW@"abba"], LyndonQ[AW@"ababb"]}
```

```
{False, True}
```

```
{AllWords[3, {"1", "2"}], AllLyndonWords[{3}, {"1", "2"}]}
```

```
{AW[111], AW[112], AW[121], AW[122], AW[211], AW[212], AW[221], AW[222]},
 {<1>, <2>, <12>, <112>, <122>}}
```

```
Table[Length[AllLyndonWords[k, {"1", "2"}]], {k, 10}]
```

```
{2, 1, 2, 3, 6, 9, 18, 30, 56, 99}
```

```
Table[Length[AllLyndonWords[k, {"1", "2", "3"}]], {k, 10}]
```

```
{3, 3, 8, 18, 48, 116, 312, 810, 2184, 5880}
```

```
BracketForm[LW["12122"]]
```

```
[[12][[12]2]]
```

```
TopBracketForm[LW["12122"]]
```

$$\overline{\overline{12} \overline{12} 2}$$

The Bracket for Lie Elements

```
b[LW["112"], LW["122"]]
```

```
<112122> + <112212>
```

```
Outer[b, AllLyndonWords[{3}, {"1", "2"}],
```

```
AllLyndonWords[{3}, {"1", "2"}]] // MatrixForm
```

$$\begin{pmatrix} 0 & \langle 12 \rangle & \langle 112 \rangle & \langle 1112 \rangle & \langle 1122 \rangle \\ -\langle 12 \rangle & 0 & -\langle 122 \rangle & -\langle 1122 \rangle & -\langle 1222 \rangle \\ -\langle 112 \rangle & \langle 122 \rangle & 0 & -\langle 11212 \rangle & \langle 12122 \rangle \\ -\langle 1112 \rangle & \langle 1122 \rangle & \langle 11212 \rangle & 0 & \langle 112122 \rangle + \langle 112212 \rangle \\ -\langle 1122 \rangle & \langle 1222 \rangle & -\langle 12122 \rangle & -\langle 112122 \rangle - \langle 112212 \rangle & 0 \end{pmatrix}$$

```

Union[Flatten[Outer[(b[#1, #2] + b[#2, #1]) &,
  AllLyndonWords[{6}, {"1", "2"}], AllLyndonWords[{6}, {"1", "2"}]
]]]
{0}

Outer[(b[#1, b[#2, #3]] + b[#2, b[#3, #1]] + b[#3, b[#1, #2]]) &,
  AllLyndonWords[{5}, {"1", "2"}],
  AllLyndonWords[{5}, {"1", "2"}], AllLyndonWords[{5}, {"1", "2"}]
] // Flatten // Union
{0}

```

LieSeries

```

Print /@ {ts1 = <"1122"> // MakeLieSeries, ts1[], ts1 /@ Range[6]};
LS[0, 0, 0]
Hold[MakeLieSeries[MakeLieSeries$351, <1122>]]
{0, 0, 0, <1122>, 0, 0}

α = RandomLieSeries[{"a", "b"}]
LS[-2 ā + 2 b̄, āb, aab̄ + 2/3 abb̄]

α@{5}
LS[-2 ā + 2 b̄, āb, aab̄ + 2/3 abb̄, -13/12 a aab̄ - 5/4 a abb̄ - 5/4 abb̄b,
-3/10 a a aab̄ - 31/24 a a abb̄ + 69/40 a abb̄b - 43/30 a b abb̄ + 7/20 a a b a b̄ + 137/120 a b b b̄]

{ts2 = <"122"> + <"11122"> // MakeLieSeries,
  ts3 = b[ts1, ts2], ts3[], ts3 /@ Range[10]}
{LS[0, 0, 122], LS[0, 0, 0], Hold[b[LS[0, 0, 0], LS[0, 0, 122]]]},
{0, 0, 0, 0, 0, 0, <1122122>, 0, -<111221122>, 0}}

{ts4 = EulerE[ts3], ts4[], ts4 /@ Range[10]}
{LS[0, 0, 0], Hold[EulerE[LS[0, 0, 0]]],
{0, 0, 0, 0, 0, 0, 7 <1122122>, 0, -9 <111221122>, 0}}

```

adPower, adSeries, and Ad

```

{xs = MakeLieSeries[LW["x"]], ys = MakeLieSeries[LW["y"]],
  ts5 = adPower[0, xs][ys], ts5[], ts5 /@ Range[5]}
{LS[x̄, 0, 0], LS[ȳ, 0, 0], LS[ȳ, 0, 0],
  Hold[adPower[0, LS[x̄, 0, 0]][LS[ȳ, 0, 0]]], {<Y>, 0, 0, 0, 0}}

```


LieMorphism

```

Print /@ {
  lm0 = LieMorphism[{LW["x"] → LW["y"]}],
  LW["x"] // lm0,
  AW["x"] // lm0,
  CW["x"] // lm0};

LieMorphism[LieMorphism$4445]

LS[ $\overline{y}$ , 0, 0]
AS[0, AW[y], 0, 0]
CWS[ $\overline{y}$ , 0, 0]

Print /@ {
  lm1 = LieMorphism[{LW["x"] → Ad[LW["y"]][LW["x"]]}],
  lm1[],
  lm1[LW["y"]],
  lm1[LW["x"]],
  lm1[LW["x"]][4],
  lm1[⟨"xxy"⟩],
  lm1[⟨"xxy"⟩][8],
  lm1[AW["x"]],
  lm1[CW["x"]]
};

LieMorphism[LieMorphism$4455]

Hold[LieMorphism[LieMorphism$4455, {⟨x⟩ → LS[ $\overline{x}$ ,  $-\overline{xy}$ ,  $\frac{1}{2}\overline{xyy}$ ]}]]]

LS[ $\overline{y}$ , 0, 0]

LS[ $\overline{x}$ ,  $-\overline{xy}$ ,  $\frac{1}{2}\overline{xyy}$ ]

-  $\frac{\langle xyxy \rangle}{6}$ 

LS[0, 0,  $\overline{xxy}$ ]

-  $\frac{\langle xxxxyyyy \rangle}{120}$  -  $\frac{\langle xyxyyyy \rangle}{30}$  -  $\frac{\langle xyxyxyyy \rangle}{24}$ 

AS[0, AW[x], -AW[xy] + AW[yx],  $\frac{AW[xyy]}{2}$  - AW[yxy] +  $\frac{AW[yyx]}{2}$ ]

CWS[ $\overline{x}$ , 0, 0]

```


AW, A Series, ι , σ

ι [BCHBase[3]]

$$\frac{AW[xxy]}{12} - \frac{AW[xyx]}{6} + \frac{AW[xyy]}{12} + \frac{AW[yxx]}{12} - \frac{AW[yxy]}{6} + \frac{AW[yyx]}{12}$$

{as = ι [BCHBase], as[5]}

Power::infy : Infinite expression $\frac{1}{0}$ encountered. >>

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. >>

$$\left\{ AS \left[\text{Indeterminate}, AW[x] + AW[y], \frac{AW[xy]}{2} - \frac{AW[yx]}{2}, \right. \right. \\ \left. \frac{AW[xxy]}{12} - \frac{AW[xyx]}{6} + \frac{AW[xyy]}{12} + \frac{AW[yxx]}{12} - \frac{AW[yxy]}{6} + \frac{AW[yyx]}{12} \right], \\ - \frac{AW[xxxxy]}{720} + \frac{AW[xxxxyx]}{180} + \frac{AW[xxxxyy]}{180} - \frac{AW[xxyxxx]}{120} - \frac{AW[xxxyxy]}{120} - \frac{AW[xxxyyx]}{120} + \\ \frac{AW[xxyyyy]}{180} + \frac{AW[xyxxx]}{180} - \frac{AW[xyxxy]}{120} + \frac{AW[xyxyx]}{30} - \frac{AW[xyxyy]}{120} - \frac{AW[xyyxx]}{120} - \\ \frac{AW[xyyxy]}{120} + \frac{AW[xyyyx]}{180} - \frac{AW[xyyy]}{720} - \frac{AW[yxxxx]}{720} + \frac{AW[yxxxxy]}{180} - \frac{AW[yxxxxy]}{120} - \\ \frac{AW[yxxyy]}{120} - \frac{AW[yxyxx]}{120} + \frac{AW[yxyxy]}{30} - \frac{AW[yxyyx]}{120} + \frac{AW[yxyyy]}{180} + \frac{AW[yyxxx]}{180} - \\ \frac{AW[yyxxx]}{120} - \frac{AW[yyxyx]}{120} - \frac{AW[yyxyy]}{120} + \frac{AW[yyxxx]}{180} + \frac{AW[yyxyy]}{180} + \frac{AW[yyyxx]}{180} - \frac{AW[yyyxy]}{720} \left. \right\}$$

{# -> σ [1, #]} & /@ AllLyndonWords[{5}, {"1", "2"}]

$$\{ \langle 1 \rangle \rightarrow AW[], \langle 2 \rangle \rightarrow 0, \langle 12 \rangle \rightarrow -AW[2], \langle 112 \rangle \rightarrow -2 AW[12] + AW[21], \langle 122 \rangle \rightarrow AW[22], \\ \langle 1112 \rangle \rightarrow -3 AW[112] + 3 AW[121] - AW[211], \langle 1122 \rangle \rightarrow 2 AW[212] - AW[221], \\ \langle 1222 \rangle \rightarrow -AW[222], \langle 11112 \rangle \rightarrow -4 AW[1112] + 6 AW[1121] - 4 AW[1211] + AW[2111], \\ \langle 11122 \rangle \rightarrow -AW[1122] + 4 AW[1212] - AW[1221] - 2 AW[2121] + AW[2211], \\ \langle 11212 \rangle \rightarrow -AW[1122] + 4 AW[1212] - AW[1221] - 3 AW[2112] + AW[2121], \\ \langle 11222 \rangle \rightarrow -2 AW[1222] + 3 AW[2122] - 3 AW[2212] + AW[2221], \\ \langle 12122 \rangle \rightarrow 2 AW[1222] - 3 AW[2122] + AW[2212], \langle 12222 \rangle \rightarrow AW[2222] \}$$

{ σ ["x", BCHBase][5], σ ["y", BCHBase][5]}

$$\left\{ - \frac{AW[yxxxxy]}{360} + \frac{AW[yxxyyx]}{240} + \frac{AW[yxxyyy]}{240} - \frac{AW[yxyxxx]}{360} - \frac{AW[yxyxyy]}{60} + \frac{AW[yxyyx]}{240} - \frac{AW[yxyyy]}{360} + \\ \frac{AW[yyxxx]}{1440} + \frac{AW[yyxyy]}{240} + \frac{AW[yyxyyx]}{240} + \frac{AW[yyxyyy]}{240} - \frac{AW[yyyxxx]}{360} - \frac{AW[yyyxyy]}{360} + \frac{AW[yyyyx]}{1440}, \\ - \frac{AW[xxxxy]}{1440} + \frac{AW[xxxxyx]}{360} + \frac{AW[xxxxyy]}{360} - \frac{AW[xxyxxx]}{240} - \frac{AW[xxxyxy]}{240} - \frac{AW[xxxyyx]}{240} - \frac{AW[xxxyyy]}{1440} + \\ \frac{AW[xyxxx]}{360} - \frac{AW[xyxyy]}{240} + \frac{AW[xyxyyx]}{60} + \frac{AW[xyxyyy]}{360} - \frac{AW[xyyxxx]}{240} - \frac{AW[xyyxyy]}{240} + \frac{AW[xyyyx]}{360} \left. \right\}$$

CW, CWSeries, tr, div

```
tr[AW["yxyxxx"]]
```

```
CW[xxyxyy]
```

```
t1 = σ["y", BCHBase] // tr
```

```
CWS[ $\left[ \frac{\widehat{x}}{2}, \frac{\widehat{xx}}{12} - \frac{\widehat{xy}}{12}, -\frac{\widehat{xyy}}{24} \right]$ ]
```

```
t1[5]
```

```
 $\frac{CW[xxxxxy]}{1440} - \frac{CW[xxxxyy]}{180} + \frac{CW[xxyxyy]}{120} + \frac{CW[xxyyyy]}{480} - \frac{CW[xyxyyy]}{720}$ 
```

```
{div["x", BCHBase][7], div["y", BCHBase][7]}
```

```
{ -  $\frac{CW[xxxxxxy]}{30240} + \frac{CW[xxxxxyy]}{2520} - \frac{CW[xxxxyxy]}{1008} - \frac{19 CW[xxxxyyy]}{15120} +$   

 $\frac{CW[xxyxyxy]}{2520} + \frac{CW[xxyxyyy]}{504} + \frac{CW[xyxyxyy]}{504} + \frac{19 CW[xyxyyyy]}{15120} +$   

 $\frac{CW[xxyxyyy]}{1680} - \frac{CW[xxyxyyy]}{280} - \frac{CW[xxyxyyy]}{504} - \frac{CW[xxyxyyy]}{1680} - \frac{CW[xxyxyyy]}{504} -$   

 $\frac{CW[xxyxyyy]}{2520} + \frac{CW[xyxyxyy]}{280} + \frac{CW[xyxyxyy]}{1008} - \frac{CW[xyxyxyy]}{2520} + \frac{CW[xxyxyyy]}{30240},$   

 $\frac{CW[xxxxxxy]}{30240} - \frac{CW[xxxxxyy]}{2520} + \frac{CW[xxxxyxy]}{1008} + \frac{19 CW[xxxxyyy]}{15120} - \frac{CW[xxyxyxy]}{2520} -$   

 $\frac{CW[xxyxyxy]}{504} - \frac{CW[xxyxyyy]}{504} - \frac{19 CW[xxyxyyy]}{15120} - \frac{CW[xxyxyyy]}{1680} +$   

 $\frac{CW[xxyxyxy]}{280} + \frac{CW[xxyxyyy]}{504} + \frac{CW[xxyxyyy]}{1680} + \frac{CW[xxyxyyy]}{504} +$   

 $\frac{CW[xxyxyyy]}{2520} - \frac{CW[xyxyxyy]}{280} - \frac{CW[xyxyxyy]}{1008} + \frac{CW[xyxyxyy]}{2520} - \frac{CW[xxyxyyy]}{30240} }$ 
```

```
t1 = MakeCWSeries[CW["xyxyyyy"]] //
```

```
LieDerivation[{LW["x"] → MakeLieSeries[b[LW["x"], LW["z"]]]}]
```

```
CWS[0, 0, 0]
```

```
t1 /@ Range[10]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, -CW[xyxyyyyzy] + CW[xyxzyyyy] - CW[xyyyyxyz] + CW[xyyyyxzy], 0, 0}
```

The Meta-Cocycle JA

```
Print /@ {y0 = LW["y"], μ0 = BCHBase,
  JA[0, y0, μ0, s],
  JA[1, y0, μ0, s],
  JA[2, y0, μ0, s],
  JA[y0, μ0]
};
```

<y>

$$\text{LS}\left[\widehat{x} + \widehat{y}, \frac{\widehat{xy}}{2}, \frac{1}{12} \widehat{xxxy} + \frac{1}{12} \widehat{xyyy}\right]$$

$$\text{CWS}\left[s \widehat{y}, \frac{s \widehat{xy}}{2} + \frac{s^2 \widehat{xy}}{2}, \frac{s \widehat{xxxy}}{12} + \frac{s^2 \widehat{xxxy}}{4} + \frac{s^3 \widehat{xxxy}}{6} - \frac{s \widehat{xyyy}}{12} - \frac{s^2 \widehat{xyyy}}{4} - \frac{s^3 \widehat{xyyy}}{6}\right]$$

$$\text{CWS}\left[s \widehat{y}, \frac{s \widehat{xy}}{2} + \frac{s^2 \widehat{xy}}{2}, \frac{s \widehat{xxxy}}{12} + \frac{s^2 \widehat{xxxy}}{4} + \frac{s^3 \widehat{xxxy}}{6} - \frac{s \widehat{xyyy}}{12} - \frac{s^2 \widehat{xyyy}}{4} - \frac{s^3 \widehat{xyyy}}{6}\right]$$

$$\text{CWS}\left[s \widehat{y}, \frac{s \widehat{xy}}{2} + \frac{s^2 \widehat{xy}}{2}, \frac{s \widehat{xxxy}}{12} + \frac{s^2 \widehat{xxxy}}{4} + \frac{s^3 \widehat{xxxy}}{6} - \frac{s \widehat{xyyy}}{12} - \frac{s^2 \widehat{xyyy}}{4} - \frac{s^3 \widehat{xyyy}}{6}\right]$$

$$\text{CWS}\left[\widehat{y}, \widehat{xy}, \frac{\widehat{xxxy}}{2} - \frac{\widehat{xyyy}}{2}\right]$$

```
$SeriesCompareDegree = $SeriesShowDegree = 8;
```

```
JA[3, y0, μ0, s] ≡ JA[4, y0, μ0, s]
```

True

```
JA[y0, μ0][6]
```

$$\frac{\text{CW}[\text{xxxxxy}]}{120} + \frac{31 \text{CW}[\text{xxxxyy}]}{48} - \frac{11 \text{CW}[\text{xxxxyxy}]}{6} + \frac{109 \text{CW}[\text{xxxxyyy}]}{36} +$$

$$\frac{7 \text{CW}[\text{xyyxxxy}]}{8} - \frac{23 \text{CW}[\text{xyyxyxy}]}{4} - \frac{23 \text{CW}[\text{xyyyxy}]}{4} + \frac{31 \text{CW}[\text{xyyyyxy}]}{48} +$$

$$\frac{28 \text{CW}[\text{xyxyxyxy}]}{3} - \frac{11 \text{CW}[\text{xyxyyyy}]}{6} + \frac{7 \text{CW}[\text{xyyyxyxy}]}{8} + \frac{\text{CW}[\text{xyyyyxy}]}{120}$$