Morgan@GRT: Quantum Hamiltonian reduction of W-algebras

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Nilpotent elements of Lie algebras & good gradings:

$\mathfrak{g}$: s.s. f.d. Lie alg. / $\mathbb{C}$ ($\mathfrak{sl}_n$ - example)

$e \in \mathfrak{g}$ nilpotent ($ad(e) \in End(\mathfrak{g})$ nilpotent)

Jacobson-Morozov: Can find h, f which make $\{h, e, f\}$ a copy of $\mathfrak{sl}_2$.

$\longrightarrow h$ is s.s., $\mathfrak{g}$ decomposes:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \qquad \mathfrak{g}_i = \{x \in \mathfrak{g} : [h, x] = ix\}$$

That's a $\mathbb{Z}$-grading of $\mathfrak{g}$.

Such gradings arising from this procedure are "Dynkin gradings"

properties:

(1) $e \in \mathfrak{g}_2$ $\qquad [h, e] = 2e$

(2) $ad_e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for $j \leq -1$

(3) $ad_e : \mathfrak{g}_j \longrightarrow \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$

__Def__ a grading is "good" for e if it satisfies 1–3.

Fact: $(1) \Rightarrow [(2) \Leftrightarrow (3)]$

Classification via "Pyramids" in type A.

$\mathfrak{g} = \mathfrak{sl}_n$, $e \sim$ jordan form, determined by a partition of n.

$\Longleftrightarrow$ dominant weights partitioning n:

$\lambda = (\lambda_1 \ldots \lambda_k)$ $\quad \lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \quad \sum \lambda_i = n$

There is a partial ordering on such by

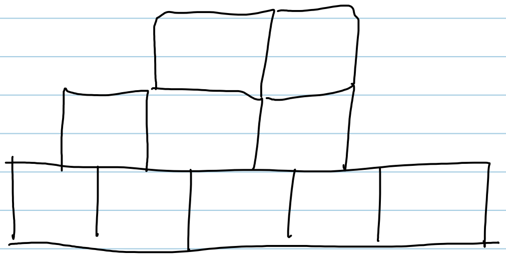
$\lambda' \geq \lambda$ if $\sum_{i=1}^{j} \lambda_i' \geq \sum_{i=1}^{j} \lambda_i \quad \forall j$

$\Longleftrightarrow$ with $\mathbb{O}_e = G \cdot e$ ← conjugation

$\mathbb{O}_e \subseteq \overline{\mathbb{O}_{e'}}$

__Def__ Given a weight $\lambda = (\lambda_1 \ldots \lambda_k)$,

make a pyramid:



λ_1

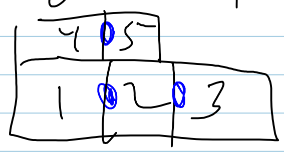
boxes have width 2.

Def a filling of a pyramid is an injection $\underline{n} \longrightarrow$ boxes of P

Thm [Elashvili–Kac] A filled pyramid of shape λ defines a nilpotent element e & a good grading for e, providing a bijection

$$\left\{\begin{matrix}\text{pyramids of}\\ \text{shape } \lambda\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{good } \mathbb{Z}\text{-gradings}\\ \text{for } e \text{ of type } \lambda\end{matrix}\right\}$$

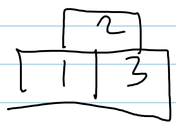
col: 0 2 4



$\longrightarrow e = (e_{12} + e_{23}) + e_{45}$

$$\deg(e_{12}) = \mathrm{col}(j) - \mathrm{col}(i)$$

Eg:



$e = e_{13}$

$$\deg e_{ij} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$

Note: $\{e_{11} - e_{33}, e_{13}, e_{31}\}$ is an $\mathfrak{sl}_2$-triple and this is its Dynkin grading.

Dynkin gradings correspond to symmetric pyramids.

Def a grading is even if all odd components vanish. Corresponds to "even pyramids".

Def Given an even good grading, set
$\mathfrak{m} = \bigoplus_{i \le -2} \mathfrak{g}_i$ "premet subalgebra"

If χ_e is the pairing w/ e under Killing
then $\chi([\mathfrak{m}, \mathfrak{m}]) = 0$ so χ
is a character of $\mathfrak{m}$.

Slodowy slices: Given $e \in \mathfrak{g}$ nilpotent, 3:43
complete to an sl_2-triple $\{h, e, f\}$

Def Slodowy slice
$$S_e = e + \ker(\mathrm{ad}(f))$$

Fact S_e is a transverse subvariety
to $\mathbb{O}_e$

Proof ....

Quick review of Hamiltonian reduction:

* Poisson variety $(X, \{\cdot, \cdot\})$
* $M \circlearrowright X$ Hamiltonian:
. . . .

* $M \circlearrowright X$ Hamiltonian:

* $\exists\ \mu : X \longrightarrow M^*$ an equivariant moment map.

select a regular value of μ s.t.

M acts freely on $\mu^{-1}(\chi)$

$X//M := \mu^{-1}(\chi)/M$ 3:57