We should have a characterization of R, not just a construction.
If R is integer-valued, it should be intrinsicly so.

KHOVANOV HOMOLOGY FOR ALTERNATING TANGLES

DROR BAR-NATAN & HERNANDO BURGOS-SOTO

ABSTRACT. We describe a "concentration on the diagonal" condition on the Khovanov complex of tangles, show that this condition is satisfied by the Khovanov complex of the single crossing tangles ($\times$) and ($\times$), and prove that it is preserved by alternating planar algebra compositions. Hence, this condition is satisfied by the Khovanov complex of all alternating tangles. Finally, in the case of 0-tangles, meaning links, our condition is equivalent to a well known result [Lee1] which states that the Khovanov homology of a non-split alternating link is supported in two lines. Thus our condition is a generalization of Lee's Theorem to the case of tangles

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1. INTRODUCTION

Khovanov [Kh] constructed an invariant of links which opened new prospects in knot theory and which is now known as the Khovanov homology. Bar-Natan in [BN1] computed this invariant and found that it is a stronger invariant than the Jones polynomial. Khovanov, Bar-Natan and Garoufalidis [Ga] formulated several conjectures related to the Khovanov homology. One of these refers to the fact that the Khovanov homology of a non-split alternating link is supported in two lines. To see this, in Table 1, we present the dimension of the groups in the Khovanov homology for the Borromean link and illustrate that the no-zero dimension groups are located in two consecutive diagonals. The fact that every alternating link satisfies this property was proved by Lee in [Lee1].

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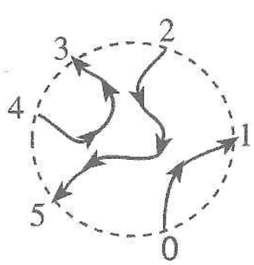
j\i	-3	-2	-1	0	1	2	3
7							1
5						2	
3						1	
1				4	2		
-1			2	4			
-3		1					
-5		2					
-7	1						

Table 1. The Khovanov homology for the Borromean link

In [BN2] Bar-Natan presented a new way of seeing the Khovanov homology. In his approach, a formal chain complex is assigned to every tangle. This formal chain complex, regarded within a special category, is an (up to homotopy) invariant of the tangle. For the particular case in which the tangle is a link, this chain complex coincides with the cube of smoothings presented in [Kh].

This local Khovanov theory was used in [BN3] to make an algorithm which provides a faster computation of the Khovanov homology of a link. The technique used in that last paper was also important for theoretical reasons. We can apply it to prove the invariance of the Khovanov homology, see [BN3]. It was also used in [BN-Mor] to give a simple proof of Lee's result stated in [Lee2], about the dimension of the Lee variant of the Khovanov homology. Here, we will show how it can be used to state a generalization to tangles of the fore-mentioned Lee's theorem [Lee1] about the Khovanov homology of alternating links. Most of the success attained by this algorithm is due to the simplification of the Khovanov complex associated to a tangle. This simplification consists in the elimination, of the loops in the smoothing of the complex (delooping), and the isomorphism in the differentials (Gaussian elimination). Indeed, given a chain complex Ω is possible apply iteratively delooping and gaussian elimination and obtain a homotopy equivalent complex with no loops and no isomorphisms. In this paper, we say that the resulting complex is a *reduced form of* Ω, and the algorithm that allows to find it will be named the *DG algorithm*.



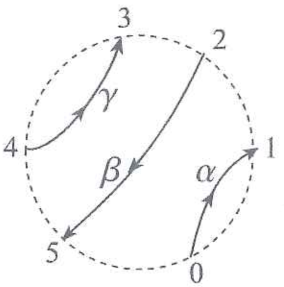
In section 7.1 we observe that the Khovanov complex of an alternating tangle can be endowed with consistent "orientations"[1], namely, every strand in every smoothing appearing in the complex can be oriented in a natural way, and likewise every cobordism, in a manner so that these orientations are consistent. (A quick glance at figures 7 and 8 on page 18 should suffice to convince the experts). Given an oriented smoothing σ, a point in the boundary of σ can be considered as an *in-boundary* point or an *out-boundary* point depending on the orientation of the respective strand in this point. We can enumerate the boundary points of the σ from 0 to $2k-1$ starting from an in-boundary point of a strand, counting counterclockwise, and finishing in the boundary point to the left of the mentioned in-boundary point. If h_α and t_α denote respectively the numbers assigned to the in-boundary

[1]Note that these are orientations of the *smoothings*, and they have nothing to do with the orientations of the components of the tangle itself

and out-boundary points of an open strand α in an oriented k-strand smoothing σ. Then the *rotation number* of α is given by: $R(\alpha) = \frac{1}{2k}[t_\alpha - h_\alpha]_{2k} - \frac{1}{2}$, where the bracket $[\ ]_{2k}$ is defined by

$$[j]_{2k} = \begin{cases} j & \text{if } j > 0 \\ j + 2k & \text{if } j < 0. \end{cases} \tag{1}$$

If α is a loop, $R(\alpha) = 1$ if α is oriented counterclockwise, and $R(\alpha) = -1$ if α is oriented clockwise. The rotation number of σ is the sum of the rotation numbers of its components (loops and strands).



$$R(\alpha) = \tfrac{1}{6} - \tfrac{1}{2} = -\tfrac{1}{3}$$
$$R(\beta) = \tfrac{1}{2} - \tfrac{1}{2} = 0$$
$$R(\gamma) = \tfrac{5}{6} - \tfrac{1}{2} = \tfrac{1}{3}$$

Figure 1. Several strands and their rotation numbers for $k = 3$.

In a manner similar to [BN2], we define a certain graded category $\mathcal{C}ob^3_o$ of *oriented cobordisms.* The objects of $\mathcal{C}ob^3_o$ are *oriented smoothing,* and the morphisms are oriented cobordisms. This category is used to define the category $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o))$ (abbreviated Kob_o) of complexes over $\mathrm{Mat}(\mathcal{C}ob^3_o)$.

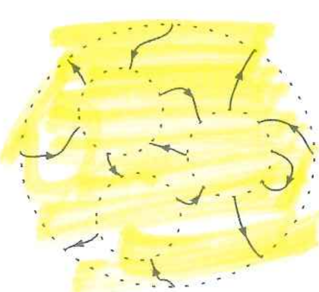
Specifically, for degree-shifted smoothings $\sigma\{q\}$, we define $R(\sigma\{q\}) := R(\sigma) + q$. We further use this *degree-shifted rotation number* to define a special class of chain complexes in Kob_o, of the form

$$\Omega : \qquad \cdots \longrightarrow \left[\sigma^r_j\right]_j \longrightarrow \left[\sigma^{r+1}_j\right]_j \longrightarrow \cdots,$$

$C(\Omega)$

which satisfies that for all degree-shifted smoothings $\sigma^r_j\{q\}$, $2r - R(\sigma^r_j\{q\})$ is a constant that we call *rotation constant* of the complex. In other words, twice the homological degrees and the degree-shifted rotation numbers of the smoothings always lie along a single diagonal. Chain complexes that have a reduced form satisfying this property are called *diagonal complexes.*

An important tool for stating our main results is the concept of *alternating planar algebra.* An alternating planar algebra is an oriented planar algebra as in [BN2, Section 5], where the d-input planar arc diagrams D satisfy the following conditions: i) The number k of strings ending on the external boundary of D is greater than 0. ii) There is complete connection among input discs of the diagram and its arcs, namely, the union of the diagram arcs and the boundary of the internal holes is a connected set. iii) The in- and out-strings alternate in every boundary component of the diagram. A planar arc diagram as



Do we need basepoints ?

this is called a type-$\mathcal{A}$ planar diagram. If Φ is an element in the planar algebra and D is a 1-input type-$\mathcal{A}$ planar diagram then $D(\Phi)$ is called a partial closure of Φ.

Using the above terminology, our main result is stated as follows:

can be reduced...

Theorem 1. *If T is an alternating non-split tangle then its Khovanov homology $Kh(T)$ is diagonal and furthermore the same is true for every partial closure of $Kh(T)$*

rough!

We say that a complex Ω is *coherently diagonal* if it is a diagonal complex whose partial closures are also diagonal. Indeed, Theorem 1 can be restated as saying that the Khovanov homology $Kh(T)$ of a non-split alternating tangle is coherently diagonal. To prove this theorem we use the fact that non-split alternating tangles form an alternative planar algebra generated for the one-crossing tangles ($\times$) and ($\times$). Thus Theorem 1 follows from the observation that $Kh(\times)$ and $Kh(\times)$ are coherently diagonal and from Theorem 2 below:

Theorem 2. *If $\Omega_1, \ldots, \Omega_n$ are coherently diagonal complexes and D is an alternating planar diagram then $D(\Omega_1, \ldots, \Omega_n)$ is coherently diagonal*

In the case of alternating tangles with no boundary, i.e., in the case of alternating links, Theorem 1 reduces to Lee's theorem on the Khovanov homology of alternating links.

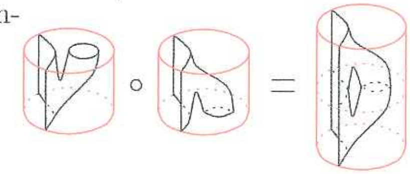
The work is organized as follows. Section 2 reviews Bar-Natan's local Khovanov theory. Section 3 is devoted to introduce the category $\mathcal{C}ob^3_o$ and gives a quick review of some concepts related to alternating planar algebras. In particular we review the concepts of rotation numbers, alternating planar diagrams, associated rotation numbers, and basic operators.

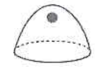
Section 4 introduces the concepts of diagonal complexes, coherently diagonal complexes, and their partial closures. We state here some results about the complexes obtained when a basic operator is applied to alternating elements. The Khovanov homology is formed from a double complex. However, when applying the DG algorithm the configuration of double complex is lost. Attempting to fix this problem in Section 5 we introduced the concept of perturbed double complex which is the object in which the DG algorithm lives. Indeed, we shall see that double complexes are special cases of perturbed complexes and that after applying any step of the DG algorithm in a perturbed complex, the complex continues being of the same class. The application of the DG algorithm leads to the proof in section 6 of Theorem 2. Finally section 7 is dedicated to the study of non-split alternating tangles. Here, we prove Theorem 1 and derive from it the Lee theorem formulated in [Lee1].

2. The local Khovanov theory: Notation and some details

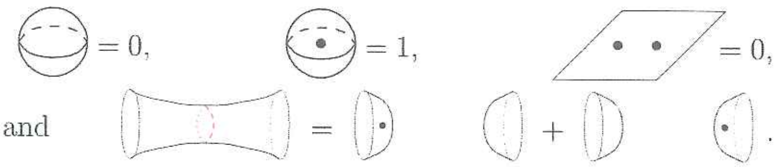
The notation and some results appearing here are treated in more details in [BN2, BN3, Naot]. Given a set B of $2k$ marked points on a circle C, a smoothing with boundary B is a union of strings $a_1, \ldots, a_n$ embedded in the planar disk for which C is the boundary, such that $\cup_{i=1}^{n} \partial a_i = B$. These strings are either closed curves, *loops*, or strings whose boundaries are points on B, *strands*. If $B = \emptyset$, the smoothing is a union of circles.

We denote $Cob^3(B)$, the category whose objects are smoothings with boundary B, and whose morphisms are cobordisms between such smoothings, regarded up to boundary preserving isotopy. The composition of morphisms is given by placing one cobordism atop the other.



Our ground ring is one in which 2^{-1} exists. The dotted figure  is used as an abbreviation of $\frac{1}{2}$  and $\mathcal{C}ob^3_{\bullet/l}(B)$ represents the category with the same objects and morphisms as $Cob^3(B)$, whose morphisms are mod out by the local relations:

(2)



We will use the notation $\mathcal{C}ob^3$ and $\mathcal{C}ob^3_{\bullet/l}$ as a generic reference, namely, $\mathcal{C}ob^3 = \bigcup_B \mathcal{C}ob^3(B)$ and $\mathcal{C}ob^3_{\bullet/l} = \bigcup_B \mathcal{C}ob^3_{\bullet/l}(B)$. If B has $2k$ elements, we usually write $\mathcal{C}ob^3_{\bullet/l}(k)$ instead of $\mathcal{C}ob^3_{\bullet/l}(B)$. If $\mathcal{C}$ is any category, $\mathrm{Mat}(\mathcal{C})$ will be the additive category whose objects are column vectors (formal direct sums) whose entries are objects of $\mathcal{C}$. Given two objects in this category,

$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \\ \vdots \\ \mathcal{O}_n \end{pmatrix} \qquad \mathcal{O}^1 = \begin{pmatrix} \mathcal{O}^1_1 \\ \mathcal{O}^1_2 \\ \vdots \\ \mathcal{O}^1_m \end{pmatrix},$$

the morphisms between these objects will be matrices whose entries are formal sums of morphisms between them. The morphisms in this additive category are added using the usual matrix addition and the morphism composition is modelled by matrix multiplication, i.e, given two appropriate morphisms $F = (F_{ik})$ and $G = (G_{kj})$ between objects of this category, then $F \circ G$ is given by

$$F \circ G = \sum_k F_{ik} G_{kj} .$$

$\mathrm{Kom}(\mathcal{C})$ will be the category of formal complexes over an additive category $\mathcal{C}$. $\mathrm{Kom}_{/h}(\mathcal{C})$ is $\mathrm{Kom}(\mathcal{C})$ modulo homotopy. We also use the abbreviations $\mathrm{Kob}(k)$ and $\mathrm{Kob}_{/h}(k)$ for denoting $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(k)))$ and $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(k)))$.

Objects and morphisms of the categories $\mathcal{C}ob^3$, $\mathcal{C}ob^3_{\bullet/l}$, $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$, $\mathrm{Kob}(k)$, and $\mathrm{Kob}_{/h}(k)$ can be seen as examples of planar algebras, i.e., if D is a n-input planar diagram, it defines an operation among elements of the previously mentioned collections. See [BN2] for specifics of how D defines operations in each of these collections. In particular, if $(\Omega_i, d_i) \in \mathrm{Kob}(k_i)$

are complexes, the complex $(\Omega, d) = D(\Omega_1, \dots, \Omega_n)$ is defined by

$$(3)\qquad \begin{aligned} \Omega^r &:= \bigoplus_{r=r_1+\cdots+r_n} D(\Omega_1^{r_1}, \dots, \Omega_n^{r_n}) \\ d|_{D(\Omega_1^{r_1},\dots,\Omega_n^{r_n})} &:= \sum_{i=1}^{n} (-1)^{\sum_{j<i} r_j} D(I_{\Omega_1^{r_1}}, \dots, d_i, \dots, I_{\Omega_n^{r_n}}), \end{aligned}$$

$D(\Omega_1, \dots, \Omega_n)$ is used here as an abbreviation of $D((\Omega_1, d_1), \dots, (\Omega_n, d_n))$.

In [BN2] the following very desirable property is also proven. The Khovanov homology is a planar algebra morphism between the planar algebras $\mathcal{T}(s)$ of oriented tangles and $\mathrm{Kob}_{/h}(k)$. That is to say, for an n-input planar diagram D, and suitable tangles $T_1, ..., T_n$, we have

$$(4)\qquad Kh(D(T_1, ..., T_n)) = D(Kh(T_1), ..., Kh(T_n)).$$

This last property is used in [BN3] to show a local algorithm for computing the Khovanov homology of a link. In that paper, Bar-Natan explained how it is possible to remove the loops in the smoothings, and some terms in the Khovanov complex $Kh(T_i)$ associated to the local tangles $T_1, ..., T_n$, and then combine them together in an n-input planar diagram D obtaining $D(Kh(T_1), ..., Kh(T_n))$, and the Khovanov homology of the original tangle.

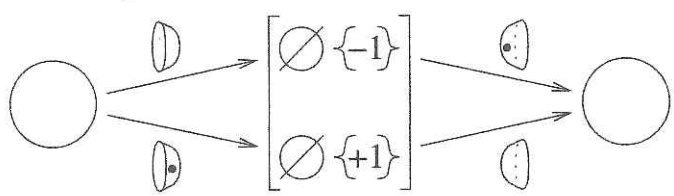
The elimination of loops and terms can be done thanks to the following: Lemma 4.1 and Lemma 4.2 in [BN3]. We copy these lemmas verbatim:

Lemma 2.1. *(Delooping) If an object S in $\mathcal{C}ob^3_{\bullet/l}$ contains a closed loop ℓ, then it is isomorphic (in* $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$*) to the direct sum of two copies $S'\{+1\}$ and $S'\{-1\}$ of S in which ℓ is removed, one taken with a degree shift of $+1$ and one with a degree shift of -1. Symbolically, this reads $\bigcirc \equiv \emptyset\{+1\} \oplus \emptyset\{-1\}$.*

The isomorphisms for the proof can be seen in:

on AltTan page 2

(5)



using all the relations in (2).

Lemma 2.2. *(Gaussian elimination, made abstract) If $\phi : b_1 \to b_2$ is an isomorphism (in some additive category $\mathcal{C}$), then the four term complex segment in* $\mathrm{Mat}(\mathcal{C})$

$$(6)\qquad \cdots\ [C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta \\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]\ \cdots$$

is isomorphic to the (direct sum) complex segment

$$(7)\qquad \cdots\ [C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]\ \cdots.$$

Both these complexes are homotopy equivalent to the (simpler) complex segment

$$\cdots\ [C] \xrightarrow{\ \ (\beta)\ \ } [D] \xrightarrow{\ \ (\epsilon-\gamma\phi^{-1}\delta)\ \ } [E] \xrightarrow{\ \ (\nu)\ \ } [F]\ \cdots . \tag{8}$$

Here C, D, E and F are arbitrary columns of objects in $\mathcal{C}$ and all Greek letters (other than ϕ) represent arbitrary matrices of morphisms in $\mathcal{C}$ (having the appropriate dimensions, domains and ranges); all matrices appearing in these complexes are block-matrices with blocks as specified. b_1 and b_2 are billed here as individual objects of $\mathcal{C}$, but they can equally well be taken to be columns of objects provided (the morphism matrix) ϕ remains invertible.

From the previous lemmas we infer that the Khovanov complex of a tangle is homotopy equivalent to a chain complex without loops in the smoothings, and in which every differential is a non-invertible cobordism. In other words, if (Ω, d) is a complex in $\mathcal{C}ob^3_{\bullet/l}$, we can use lemmas 2.1, 2.2, and obtain a homotopy equivalent chain complex (Ω', d') with no loop in its smoothings and no invertible cobordism in its differentials. This complex (Ω', d') is what we call a *reduced form* of (Ω, d) .

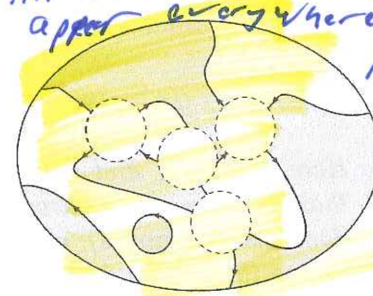
We say that a complex that has these two properties is "reduced".

3. The category Kob_o and alternating planar algebras

In this section we introduce an alternating orientation in the objects of $\mathcal{C}ob^3_{\bullet/l}(k)$. This orientation induces an orientation in the cobordisms of this category. These oriented k-strand smoothings and cobordisms form the objects and morphisms in a new category. The composition between cobordisms in this oriented category is defined in the standard way, and it is regarded as a graded category, in the sense of [BN2, Section 6]. We subject out the cobordisms in this oriented category to the relations in (2) and denote it as $\mathcal{C}ob^3_o(k)$. Now we can follow [BN2] and define sequentially the categories, $\mathrm{Mat}(\mathcal{C}ob^3_o(k))$, $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$ and $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$. These last two categories are what we denote $\mathrm{Kob}_o(k)$, and $\mathrm{Kob}_{o/h}$. As usual, we use Kob_o, and $\mathrm{Kob}_{o/h}$, to denote $\bigcup_k \mathrm{Kob}_o(k)$ and $\bigcup_k \mathrm{Kob}_{o/h}(k)$ respectively.

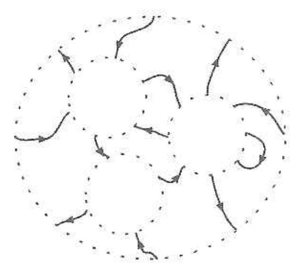
The orientation in the smoothings is done in such a way that the orientation in the strands is alternating in the boundary of the disc where they are embedded. That can be achieved by shading the connected components of the complement of the smoothing (*regions*) in the disc black and white, so that regions with common boundary have different shadings. This checkerboard coloring defines an alternating orientation of the strings of the smoothing. We simply draw arrows on the strings so that the arrows point in a counterclockwise direction around the black regions. We can forget the coloring and consider that the regions are divided into: *negative*, those with boundary oriented clockwise; and *positive*, those with boundary oriented counterclockwise. This orientation in the smoothings and the rotation number associated to it were previously utilized in [Bur] to generalize a Thistlethwaite's result for the Jones polynomial stated in [Th].

If the shading is added information, it should appear everywhere. otherwise perhaps remove



After removing loops, the resulting collection of alternating oriented objects obtained will be denoted with the symbol $\mathcal{S}_o$. A d-input planar diagram with an alternating orientation of its arcs, which could be also achieved by a checkerboard coloring of the disc, provides a good tool for the horizontal composition of objects in $\mathcal{S}_o$. Given smoothings $\sigma_1, ..., \sigma_d$, a suitable alternating d-input planar diagram D to compose them has the property that the i-th input disc has as many boundary arc points as σ_i. Moreover placing σ_i in the i-th input disc, the orientation (the coloring) of σ_i and D match. An alternatively oriented d-input planar diagram as this, provides a good tool for the horizontal composition of objects not only in $\mathcal{S}_o$, but also in $\mathcal{C}ob^3_o$, $\mathrm{Mat}(\mathcal{C}ob^3_o(k))$, Kob_o, and $\mathrm{Kob}_{o/h}$. For the horizontal composition of morphisms in $\mathcal{C}ob^3_o$, the coloring of the disk, define a colouring in the vertical cylinder $D\times[0,1]$ in such a way that vertical curtains separate solid regions of different colour.

We are going to use these alternating diagrams to compute non-split alternating tangles, and we want to preserve the non-split property of the tangle. Hence, it will be better if we use d-input type $\mathcal{A}$ diagrams.



A d-input type-$\mathcal{A}$ diagram has an even number of strings ending in each of its boundary components, and every string that begins in the external boundary ends in a boundary of an internal disk. We can classify the strings as: *curls*, if they have its ends in the same input disc; *interconnecting arcs*, if its ends are in different input discs, and *boundary arcs*, if they have one end in an input disc and the other in the external boundary of the output disc. The arcs and the boundaries of the discs divide the surface of the diagram into disjoint regions. Some arcs and regions will be useful in the following definitions and propositions.

Definition 3.1. We assign the following numbers to every d-input planar diagram D:

- i_D: number of interconnecting arcs and curls, i.e., the number of non-boundary arcs.
- w_D: number of negative internal regions. That is, in the checkerboard coloring, the white regions whose boundary does not meet the external boundary of D.
- R_D: the rotation associated number, which is given by the formula

$$R_D = \frac{1}{2}(1 + i_D - d) - w_D$$

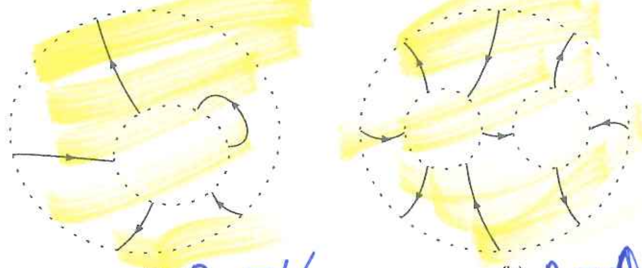
with $d:=n$.

For input-count we sometimes use d & sometimes n.

Proposition 3.2. *Given the smoothings $\sigma_1, ..., \sigma_d$ and a suitable d-input planar diagram D, where every smoothing can be placed, the rotation number of $D(\sigma_1, ..., \sigma_d)$ is:*

$$R(D(\sigma_1, ..., \sigma_d)) = R_D + \sum_{i=1}^{d} R(\sigma_i) \tag{9}$$

Diagrams with only one or two input discs deserves special attention. Operators defined from diagrams like these are very important for our purposes since some of them are considered as the generators of the entire collection of operators in a connected alternating planar algebra.



(a) $R_D = \mp 1/2$ (b) $R_D = 0$

Figure 2. Examples of basic planar diagrams

Definition 3.3. A basic planar diagram (See Figure 2) is a 1-input alternating planar diagram with a curl in it, or a 2-input alternating planar diagram with only one interconnecting arc. A basic operator is one defined from a basic planar diagram. A negative unary basic operator is one defined from a basic 1-input diagram where the curl completes a negative loop. A positive unary basic operator is one defined from a basic 1-input diagram where the curl completes a positive loop. A binary operator is one defined from a basic 2-input planar diagram.

Proposition 3.4. *The rotation associated number of a planar diagram belongs to $\frac{1}{2}\mathbb{Z}$ and the case when we have a basic planar diagram it is given as follows:*

- *If D is a negative unary basic operator, $R_D = -\frac{1}{2}$*
- *If D is a binary basic operator, $R_D = 0$*
- *If D is a positive unary basic operator, $R_D = \frac{1}{2}$*

Proposition 3.5. *Any operator D in an alternatively oriented planar algebra is the finite composition of basic operators.*

4. Diagonal complexes

Once we have applied lemma 2.1 to an element of Kob_o, we obtain a complex (Ω, d) which preserves some properties of the former one, but with a change in the rotation number of the element $\sigma\{q_\sigma\}$, in which we have applied the delooping. Indeed, the smoothing has been replaced in the complex by a couple whose rotation number has changed either by -1 or by +1. This shift in the rotation number could be even greater if we continue removing loops in the same smoothing. We know, from Lemma 2.1, that there is also a change in the grading shift of the smoothings. Thus, it would be prudent to define a concept that states a relation between the rotation number of σ and its grading shift q_σ.

Definition 4.1. Let (Ω, d) be a class-representative of $\mathrm{Kob}_{o/h}$, and let $\sigma_i\{q_i\}$ be a shifted degree object in Ω^r, then its degree-shifted rotation number is $R(\sigma_i\{q_i\}) = R(\sigma_i) + q_i$

Definition 4.2. A degree-preserving differential chain complex in $\mathrm{Kob}_{o/h}$ is called a diagonal complex if it has a reduced form (Ω, d)

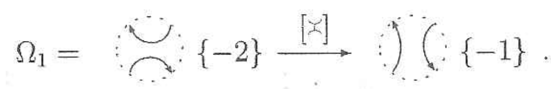
$$\cdots \Omega^r \xrightarrow{d^r} \Omega^{r+1} \cdots$$

in Kob_o, satisfying that for each homological degree r, and each shifted degree object $\sigma_i\{q_i\}$ in Ω^r, $2r - R(\sigma_i\{q_i\}) = C_\Omega$, where C_Ω is a constant that we call rotation constant of (Ω, d).

The following are some examples of diagonal complexes in Kob_o.

Example 4.3. As in [BN3], a dotted line represent a dotted curtain, and)(represents the saddle)(⟶ ≍

(1)



This is the Khovanov homology of the negative crossing X, now with orientation in the smoothings. Remember that the first term has homological degree -1. In this example the rotation number in the first term is $-\frac{1}{2}$ and in the second term it is $\frac{1}{2}$. Observe that in each case, the difference between 2 times the homological degree r and the shifted rotation number is $\frac{1}{2}$.

I'm not sure this is needed. On the other hand, we need examples of c.d. that are not d. and of d. that are not c.d.

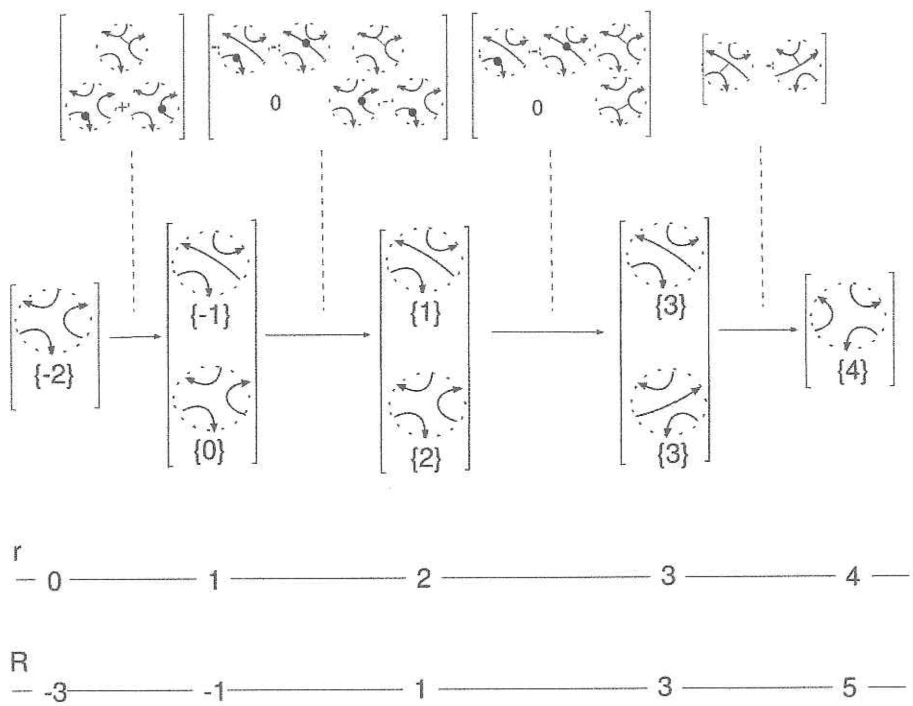


Figure 3. A diagonal complex.

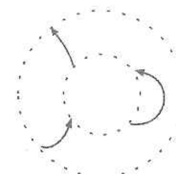
(2) In Figure 3, the number below each smoothing is the grading shift of the smoothing. The upper line below the complex represents the homological degree r, and the lower one represents the degree-shifted rotation number. For instance, the rotation number in the first smoothing with homological degree 1 has rotation number 0 and a grading shift by -1. In the second smoothing of the same vector, the rotation number is -1 and its grading shift is 0, so both terms have the same degree-shifted rotation number. We see in this example, that for each r we have that $2r - R = 3$, so this is a diagonal complex with rotation constant 3.

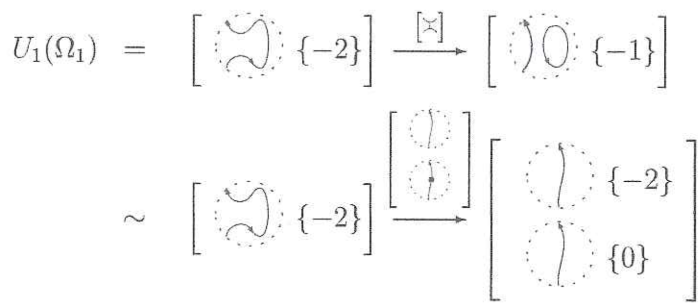
4.1. **Applying unary operators.** The reduced complexes in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o))$ can be inserted in appropriate unary basic planar diagrams, and then apply the DG algorithm to obtain again a reduced complex in Kob_o. The whole process can be summarized in the following steps:

(1) placing of the complex in the corresponding input disc of the d-input planar arc diagram by using equations (3),
(2) removing the loops obtained by applying lemma 2.1, i.e, replacing each of them by a copy of $\emptyset\{+1\} \oplus \emptyset\{-1\}$, and
(3) applying gaussian elimination (lemma 2.2), and removing in this way each invertible differential in the complex.

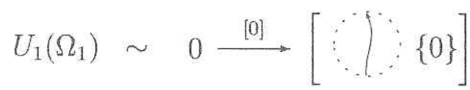
Definition 4.4. Let (Ω, d) be a chain complex in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$, then a partial closure of (Ω, d) is a chain complex of the form $D_l \circ \cdots \circ D_1(\Omega)$ where $0 \leq l < k$ and every D_i $(1 \leq i \leq l)$ is a unary basic operator.

We have diagonal complexes whose partial closures are again diagonal complexes. For instance, embedding Ω_1 of the example 4.3 in a unary basic planar diagram U_1 as the one on the right which has an associated rotation number $R_{U_1} = \frac{1}{2}$, produces the chain complex.





The last complex is the result of applying lemma 2.1. Now applying lemma 2.2, we obtain a homotopy equivalent complex



which is also a diagonal complex, but now with rotation constant zero.

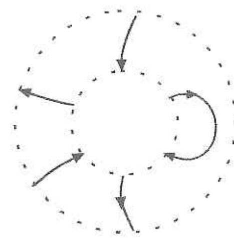
$C(D_u)$? C_Ω?

Definition 4.5. Let (Ω, d) be a diagonal complex in Kob_o with rotation constant C_R. We say that (Ω, d) is *coherently diagonal,* if for any appropriated unary operator with associated rotation number R_U, the closure $U(\Omega, d)$ has a reduced form which is a diagonal complex with rotation constant $C_R - R_U$.

We denote as $\mathcal{D}(k)$ the collection of all coherently diagonal complexes in $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$, and as usual, we write $\mathcal{D}$ to denote $\bigcup_k \mathcal{D}(k)$. It is easy to prove that any coherently diagonal complex satisfies that:

(1) after delooping any of the positive loops obtained in any of its partial closure, by using lemma 2.2, the negative shifted-degree term can be eliminated.

(2) after delooping any of the negative loops obtained in any of its partial closure, by using lemma 2.2, the positive shifted-degree term can be eliminated.



Since the computation of any other of its partial closures produces other diagonal complex, the complex Ω_1 of the example 4.3 is an element of $\mathcal{D}(2)$. Another example of coherently diagonal complex is the complex Ω_2 of the same example. This last complex has $C_R = 3$. All of its partial closures $U(\Omega_2)$ are diagonal complexes with rotation constants given by $3-R_U$. Here, we only calculate the one produced by inserting the element in the closure disc U, with $R_U = -\frac{1}{2}$, that appears on the right. It will be easy for the reader to compute the other partial closures. Inserting Ω_2 in U produces the complex of Figure 4, which is also a diagonal complex, but with a loop in some of its

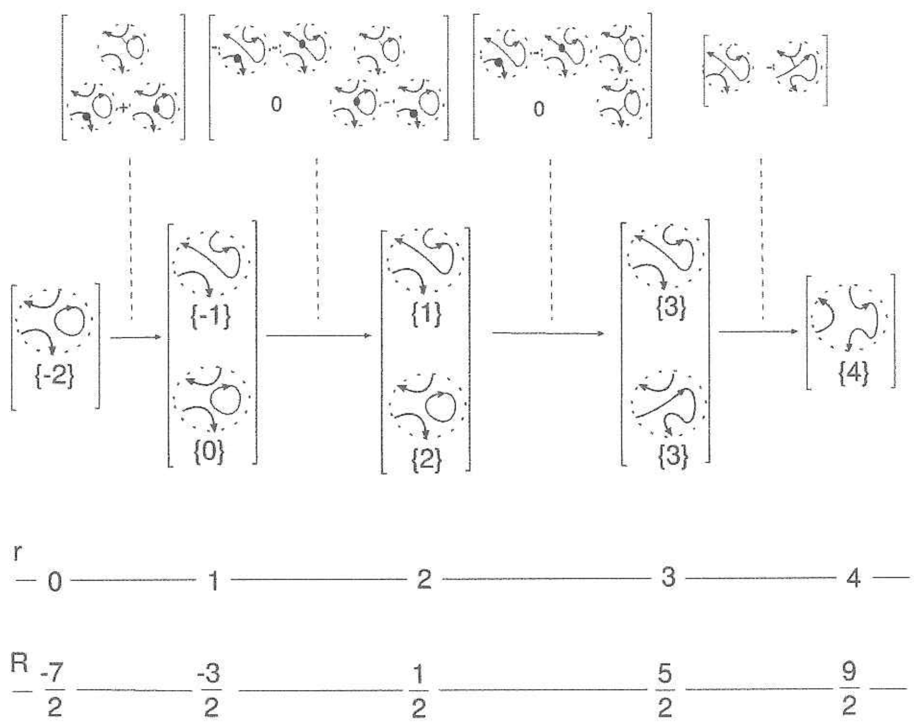


Figure 4. A diagonal complex inserted in a negative unary basic diagram U.

smoothings. We observe that the rotation number of the smoothings have decreased in $\frac{1}{2}$ after having been inserted in a negative unary basic diagram.
After applying the DG algorithm, we obtain the complex in Figure 5 which is also a diagonal complex, but now with rotation constant $\frac{7}{2}$.

4.2. Applying binary operators.

Proposition 4.6. *If D is an appropriate binary basic operator and (Ψ, e), (Φ, f) are diagonal complexes in* Kob_o *with rotation constants C_Ψ and C_Φ respectively, then $D(\Psi, \Phi)$ is a diagonal complex with rotation constant $C_\Psi + C_\Phi$.*

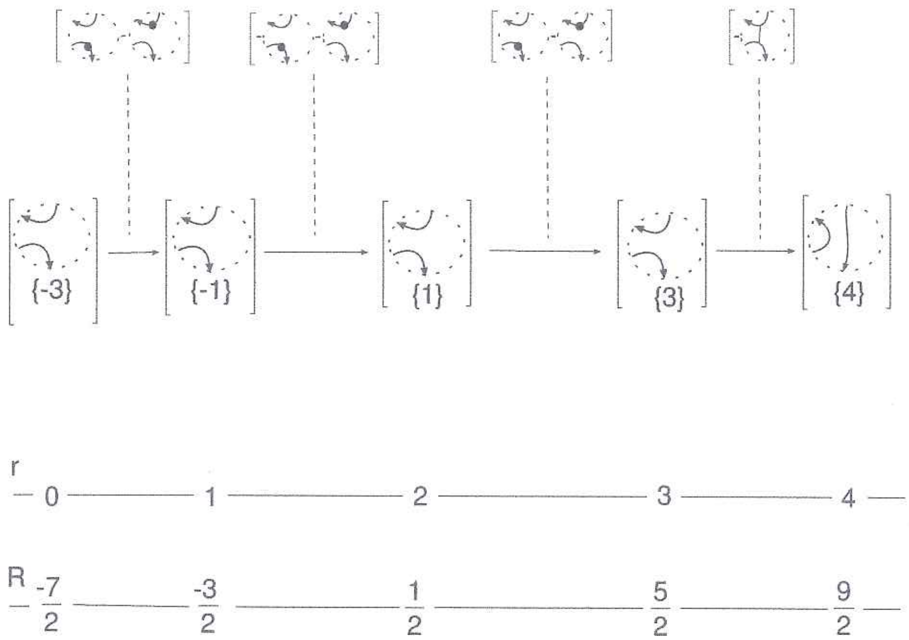


Figure 5. A partial closure of a coherently diagonal complex is also a diagonal complex.

Proof. Inserting (Ψ, e) and (Φ, f) in the disc D produces the complex $(\Omega, d) = D(\Psi, \Phi)$, which by equation (3) satisfies

$$\Omega^r = \bigoplus_{r=s+t} D(\Psi^s, \Phi^t) \tag{10}$$

and

$$d_{|D(\Psi^s,\Phi^t)} = D(e, I_{\Phi^t}) + (-1)^s D(I_{\Psi^s}, f) \tag{11}$$

If $\psi\{q_\psi\}$ and $\phi\{q_\phi\}$ are respectively elements in the vectors Ψ^s and Φ^t, so by equation (10) the elements in the vector Ω^r are of the form $D(\psi, \phi)\{q_\psi + q_\phi\}$. As ψ and ϕ are smoothings with no loops, the same we have for $D(\psi, \phi)$ and by using propositions 3.2 and 3.4, we obtain

$$R(D(\psi, \phi)) + q_\psi + q_\phi = R(\psi) + R(\phi) + q_\psi + q_\phi.$$

Therefore, the homological degree r satisfies

$$2r - R(D(\psi, \phi)) = s - (R(\psi) + q_\psi) + t - (R(\phi) + q_\phi) = C_\Psi + C_\Phi$$

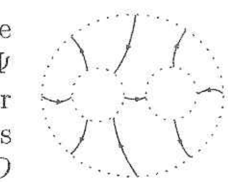
□

I'm not sure if prop. 4.7 & 4.8 are necessary.

Proposition 4.7. *Let (Ψ, e) and (Φ, f) be complex es in $\mathcal{D}$ with rotation constant C_Ψ and C_Φ respectively, and let D be a binary basic planar operator ~~in~~ for which $D(\Psi, \Phi)$ is well defined. For each partial closure $C(D(\Psi, \Phi))$, there exists an operator D' defined ~~on~~ using? a diagram without curls and chain complexes Ψ', Φ' in $\mathcal{D}$ such that*

$$C(D(\Psi, \Phi)) = D'(\Psi', \Phi')$$

Any statement about the rotation numbers of Φ' & Ψ'?



Proof. We have a binary basic planar diagram D as the one at the right. A closure of $D(\Psi, \Phi)$ can be regarded as the composition of Ψ and Φ in an operator $C(D)$ defined from this closure, i.e., an operator defined from a disc D embedded in a closure disc. Consider the strings with ends in the same input disc (the curls of the diagram). Since D is a binary basic operator, in each input disc there is at least one string that is not a curl. Thus, we can regard the disc D as a composition of two closure disc E, E' embedded in a binary planar diagram D' with no curl such that $C(D) = D(E, E')$. See Figure 6. Hence, $C(D(\Psi, \Phi)) = D(E(\Psi), E'(\Phi))$. Since $E(\Psi)$ and $E'(\Phi)$ are respectively closures of Ψ and Φ which are elements of $\mathcal{D}$, the proposition has been proved. □

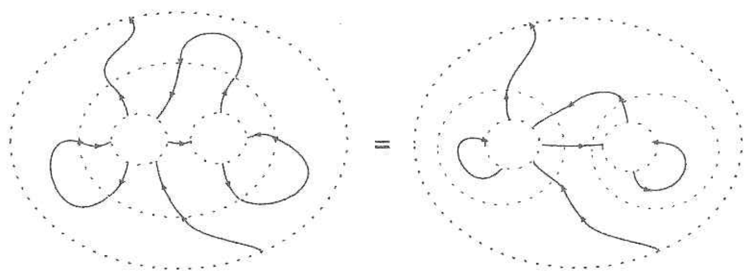


Figure 6. The closure of a binary operator can be ~~considerate~~ considered as the binary composition of two unary operator closures

Proposition 4.8. *Let σ and τ be smoothings, and let D be a suitable binary planar operator defined from a no-curl planar arc diagram with output disc D_0, input discs D_1, D_2, associated rotation constant R_D and with at least one boundary arc ending in D_1. Then there exists a closure operator C and a unary operator D' defined from a no-curl planar arc diagram such that $D(\sigma, \tau) = D'(C(\sigma))$. Moreover, if $(\Omega, d) \in \mathcal{D}$ has rotation constant C_Ω, then $D(\Omega, \tau)$ is a diagonal complex with rotation constant $C_\Omega - R(\tau) - R_D$.*

Proof. To prove that the rotation constant of $D(\Omega, \tau)$ is $C_\Omega - R(\tau) - R_D$, we observe that for each smoothing $\sigma\{q_\sigma\}$ in Ω the shifted rotation number satisfies $R(D(\sigma\{q_\sigma\}, \tau)) = R_D + R(\sigma\{q_\sigma\}) + R(\tau) = R_D + 2r - C_\Omega + R(\tau)$. Therefore, $2r - R(D(\sigma\{q_\sigma\}, \tau)) = C_\Omega - R(\tau) - R_D$ □

5. PERTURBED DOUBLE COMPLEXES

Given an additive category $\mathcal{C}$, an *(upward) perturbed double complex* in $\mathcal{C}$ is a family Ω of objects $\{\Omega_{p,q}\}$ of $\mathcal{C}$ indexed in $\mathbb{Z} \times \mathbb{Z}$, together with morphisms

=

$$d^i : \Omega_{p,q} \to \Omega_{p-i+1,q+i} \text{ for each } i \geq 0,$$

such that if $\mathbf{d} : \sum d^i$ then $\mathbf{d}^2 = 0$; or alternatively,

$$\sum_{i=0}^{k} d^i \circ d^{k-i} = 0 \text{ for each } k \geq 0 \tag{12}$$

It will be convenient to illustrate the perturbed double complex as a lattice in which any node $\Omega_{p,q}$ is the domain of ~~an infinite number of~~ arrows $d^0, d^1, \ldots$, which

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \Omega_{p-2,q} & \Omega_{p-2,q+1} & \Omega_{p-2,q+2} & \Omega_{p-2,q+3} & \cdots \\
\cdots & \Omega_{p-1,q} & \Omega_{p-1,q+1} & \Omega_{p-1,q+2} & \Omega_{p-1,q+3} & \cdots \\
\cdots & \Omega_{p,q} \arrow[r, "d^1"'] \arrow[d, "d^0"] \arrow[ru, "d^2"'] \arrow[rrruu, "d^3"] & \Omega_{p,q+1} & \Omega_{p,q+2} & \Omega_{p,q+3} & \cdots \\
\cdots & \Omega_{p+1,q} & \Omega_{p+1,q+1} & \Omega_{p+1,q+2} & \Omega_{p+1,q+3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

satisfies the following infinite number of conditions

For $k = 0$: Equation (12) reduces to $d^0 \circ d^0 = 0$. This condition is equivalent to saying that for each fixed $q \in \mathbb{Z}$, the objects $\Omega_{p,q}$ and the morphisms $d^0 : \Omega_{p,q} \to \Omega_{p+1,q}$ form a complex. We call these complexes the *vertical complexes* $\Omega_{\bullet,q}$ of the reticular complex

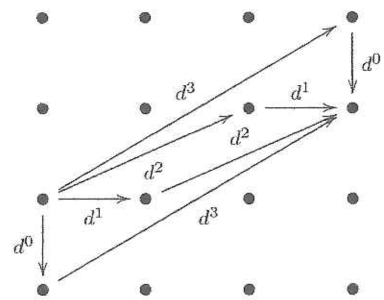
For $k = 1$: Equation (12) reduces to $d^0 \circ d^1 + d^1 \circ d^0 = 0$ This condition is equivalent to say that all the squares in the diagram anticommute

$$
\begin{tikzcd}
 & \vdots \arrow[d, "d^0"] & \vdots \arrow[d, "d^0"] & \\
\cdots \arrow[r, "d^1"] & \bullet \arrow[r, "d^1"] \arrow[d, "d^0"] & \bullet \arrow[r, "d^1"] \arrow[d, "d^0"] & \cdots \\
\cdots \arrow[r, "d^1"] & \bullet \arrow[r, "d^1"] \arrow[d, "d^0"] & \bullet \arrow[r, "d^1"] \arrow[d, "d^0"] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}
$$

For $k = 2$: Equation (12) reduces to $d^0 \circ d^2 + d^1 \circ d^1 + d^0 \circ d^2 = 0$. This states that for each p, q the sum of $d^1 \circ d^1$ plus the compositions along consecutive sides of the parallelogram with vertices on $\Omega_{p,q}$, $\Omega_{p-1,q-2}$, $\Omega_{p+1,q}$ and $\Omega_{p,q+2}$ is zero.

$$
\begin{tikzcd}
\bullet & \bullet & \bullet \arrow[d, "d^0"] \\
\bullet \arrow[r, "d^1"'] \arrow[rru, "d^2"] \arrow[d, "d^0"'] & \bullet \arrow[r, "d^1"] & \bullet \\
\bullet \arrow[rru, "d^2"'] & \bullet & \bullet
\end{tikzcd}
$$

For any $k \geq 0$: Equation (12) states that the sum of the compositions along consecutive sides of all possible *parallelograms* with diagonal on $\Omega_{p,q}$ and $\Omega_{p-k,q+k+2}$ is zero.



We must include in the sum, a composition $d^{k/2} \circ d^{k/2}$ along the common diagonal of the parallelograms, for cases where k is an even integer.

In the same way as in the case of double complexes, a perturbed double complex is associated to a chain complex that we denominate its *total complex*, abbreviated $\mathrm{Tot}(\Omega)$, and defined as follows:

Definition 5.1. Given a perturbed double complex Ω, its total complex $\mathrm{Tot}(\Omega)$ is defined by

$$\mathrm{Tot}(\Omega)^n = \bigoplus_{p+q=n} \Omega_{p,q}$$

$$\mathbf{d}\big|_{\Omega_{p,q}} = \sum_{i\geq 0} d^i$$

Note that the condition stated by equation (12) makes certain that $\mathrm{Tot}(\Omega)$ is indeed a chain complex. We observe also that double complexes are just the special cases of perturbed double complexes in which $d^i = 0$ for each $i \geq 2$.

If no confusion arises, from now on we omit specific mention of the adjective total and we will write just Ω when we refer to $\mathrm{Tot}(\Omega)$. We shall simply say "perturbed double complex" to mean the total complex associated to it.

One desired feature of perturbed double complex Ω is that the DG algorithm works well when applied to one of its vertical complexes $\Omega_{\bullet,q}$. What we mean with this last sentence is that the homotopy equivalent complex obtained after applying the DG algorithm in objects and morphisms located in the same vertical complex of a perturbed double complex is itself a perturbed double complex. We see this inmmediately.

First of all, by applying Lemma 2.1 in $\Omega_{p,q}$, we do not change the configuration of perturbed double complex. Indeed, if $f : \Omega_{p,q} \to \Omega'_{p,q}$ is an isomorphism, then it is possible to obtain a perturbed double complex Ω' homotopy equivalent to Ω by substituting $\Omega_{p,q}$ by $\Omega'_{p,q}$, and by replacing any morphism d^i with image in $\Omega_{p,q}$ by the morphism $f \circ d^i$, and any morphism d^j with domain in $\Omega_{p,q}$ by $d^j \circ f^{-1}$.

Secondly, if $\phi : b_1 \to b_2$ is an isomorphism in $\mathcal{C}$, and if D_0, E_0 are column vectors of object in $\mathcal{C}$. Given a perturbed double complex Ω with

$$\Omega_{p,q} = \begin{bmatrix} b_1 \\ D_0 \end{bmatrix} \text{ and } \Omega_{p+1,q} = \begin{bmatrix} b_2 \\ E_0 \end{bmatrix},$$

then eliminating ϕ by applying Lemma 2.2 does not bring any change in a vertical chain $\Omega_{\bullet,r}$ with $r \neq q$. Moreover, since the application of this lemma does not bring any new type of arrow in Ω, we have that the homotopy equivalent complex obtained is also a perturbed double complex. Hence, the DG algorithm can be applied to a vertical complex in Ω in such a way that the others vertical complexes remain unchanged.

6. Proof of Theorem 2

Before proving the first main theorem, let us state the following result.

Proposition 6.1. *Let (Ω, d) be a coherently diagonal complex with rotation constants C_Ω. Let $[\sigma_j]_j$ be a vector of degree-shifted smoothings all of them with the same rotation number R. Suppose that D is a binary operator with associated rotation constant R_D and at least one boundary arc coming from the first input disc. Then $D(\Omega, [\sigma_j]_j)$ has a reduced diagonal complex with rotation constant $C_\Omega - R - R_D$.*

Proof. The complex $D(\Omega, [\sigma_j]_j)$ is the direct sum $\oplus_j [D(\Omega, \sigma_j)]$. Thus, the proposition follows from the observation that by proposition 4.8, each of its direct summands $D(\Omega, \sigma_j)$ is a coherently diagonal complex with rotation constant $C_\Omega - R - R_D$. □

Picture 2.

Lemma 6.2. *Let $(\Omega_1, \mathbf{d}_1)$ be a coherently diagonal complex with rotation constants C_1. Let $(\Omega_2, \mathbf{d}_2)$ a diagonal complex with rotation constant C_2. Suppose that D is a binary operator with associated rotation constant R_D and at least one boundary arc coming from the first input disc. Then $D(\Omega_1, \Omega_2)$ has a reduced diagonal complex with rotation constant $C_1 + C_2 - R_D$.*

"The Main Lemma"

Proof. Observe that $\Omega = D(\Omega_1, \Omega_2)$ is a double complex. Indeed, if Ω_2 is the chain complex

$$\cdots \longrightarrow \Omega_2^{q-1} \longrightarrow \Omega_2^q \longrightarrow \Omega_2^{q+1} \cdots$$

then $\Omega_{\bullet,q}$ is the planar composition $D(\Omega_1, \Omega_2^q)$. Since Ω_2 is a diagonal complex, any of the smoothings in Ω_2^q has the same rotation number, $2q - C_2$. Thus, by proposition 6.1, $\Omega_{\bullet,q}$ is homotopy equivalent to a reduced diagonal complex $\Omega'_{\bullet,q}$ with rotation constant $C_1 + C_2 - 2q - R_D$. We already know that we can apply delooping and gaussian elimination in Ω involving only elements of $\Omega_{\bullet,q}$ and obtain a homotopy equivalent complex that has no changes in another vertical chain complex of Ω. In consequence, Ω is homotopy equivalent to a perturbed complex Ω' in which each $\Omega_{\bullet,q}$ has been replaced by its correspondent reduced complex $\Omega'_{\bullet,q}$. Thus, for each obtained $\Omega'_{\bullet,q}$ and each of its homological degree p, we have $2p - R(\Omega_{p,q}) = C_1 + C_2 - 2q - R_D$. Therefore, Ω' is a diagonal complex with rotation constant $C_1 + C_2 - R_D$. □

Proof. (Of **Theorem 2**) By proposition 3.5, we only need to prove that $\mathcal{D}$ is closed under composition of basic operators. Let $(\Omega_1, d_1) \in \mathcal{D}$ and let U be a basic unary operator. Since $U(\Omega_1)$ is a partial closure of (Ω_1, d_1), $U(\Omega_1)$ is diagonal. Furthermore any partial closure of $U(\Omega_1)$ is also a partial closure of (Ω_1, d_1), so $U(\Omega_1) \in \mathcal{D}$.
Let (Ω_1, d_1) and (Ω_2, d_2) be elements of $\mathcal{D}$, and let D be a basic binary operator. Since D is defined from a type-$\mathcal{A}$ diagram, there is at least one boundary arc in $\mathcal{D}$. Without loss of generality, we can assume that there is one boundary arc ending in the first input disc of D. By proposition 4.6, $D(\Omega_1, \Omega_2)$ is a diagonal complex. Let $C(D(\Omega_1, \Omega_2))$ be a partial closure of $D(\Omega_1, \Omega_2)$, by proposition 4.7 there exist $\Omega'_1, \Omega'_2 \in \mathcal{D}$ and a binary operator D' defined from a no-curl planar diagram such that $C(D(\Omega_1, \Omega_2)) = D'(\Omega'_1, \Omega'_2)$. By using Lemma 6.2, we obtain that $D'(\Omega'_1, \Omega'_2)$ is a diagonal complex. □

7. Non-split alternating tangles and Lee's theorem

7.1. Gravity information. Given a diagram of an alternating tangle, we add to it some special information which will help us to compose the Khovanov invariant of an alternating tangle in an alternating planar diagram. This is illustrated by drawing, in every strand of the diagram, an arrow pointing in to the undercrossing, or equivalently (if we have alternation), pointing out from the overcrossing. In a neighborhood of a crossing, the diagram looks like the one in Figure 7(a). Figure 7(b) shows a diagram of a tangle in which we have added the

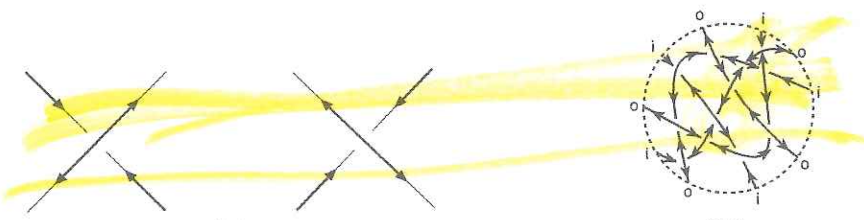


Figure 7. (a) Gravity information in a neighbourhood of a crossing. (b) Gravity information in the tangle. We use: o for out-boundary points and i for in-boundary points.

gravity information to the whole tangle. We observe, (see Figure 7(a)) that if we make a smoothing in the crossing, the orientation provided by the gravity information is preserved, and that a 0-smoothing is clockwise and 1-smoothing is counterclockwise, see figure 8. It is easily observed as well that if we go into a non-split alternating tangle for an in-boundary point and turn to the right (a 0-smoothing) every time that we meet a crossing, we are going to get out of the tangle along the boundary point immediately to the right. Hence, the in- and out-boundary points of the diagram of the tangle are arranged alternatingly. These two observations are stated in the following two propositions:

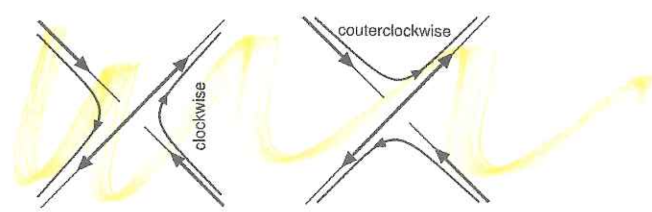


Figure 8. The smoothings in the diagrams preserve the gravity information

Proposition 7.1. *The 0-smoothings and 1-smoothings preserve the gravity information. The first ones provides a clockwise orientation of the pair of strands in the smoothing, and the last provides a counterclockwise orientation.*

Proposition 7.2. *In any non-split alternating tangle, if the k-th boundary point is an in-boundary point, then the* $(k+1)$*-th boundary point is an out-boundary point.*

Propositions 7.1 and 7.2 indicate that the smoothings of a tangle could be drawn as trivial tangles in which arcs are oriented alternatingly. Therefore, the Khovanov homology produces an alternating planar algebra morphism in the sense of [BN2].

7.2. **Proof of Theorem 1.** This proof is a direct application of Theorem 2, and the fact that the Khovanov homology is an alternating planar algebras morphism

space

This is split on several pages & should be more prominent.

Proof. (Of **Theorem 1**)The Khovanov complex of a 1-crossing tangle is an element of $\mathcal{D}(2)$. See first example 4.3. Any non-split alternating k-strand tangle with n crossing T, is obtained by a composition of n of these 1-crossing tangles, T_1,...,T_n, in a n-input type-$\mathcal{A}$ planar diagram. Since the Khovanov homology is a planar algebra morphism, by using the same n-input planar diagram for composing $Kh(T_1)$,...,$Kh(T_n)$ we obtain the Khovanov homology of the original tangle. According to Theorem 1, this is a complex in $\mathcal{D}$. □

Corollary 7.3. *The Khovanov complex $[T]$ of a a non-split alternating 1-tangle T is homotopy equivalent to a complex*

$$\cdots \longrightarrow \Omega^r\{2r+K\} \longrightarrow \Omega^{r+1}\{2(r+1)+K\} \longrightarrow \cdots$$

where every Ω^r is a vector of single lines, and K is a constant.

Proof. We only have to apply theorem 1 and see that the rotation number of a 1 open arc, which is the only simple possible smoothing resulting from a 1-tangle, is zero. □

Figure 9 shows a diagonal complex which is obtained from embedding the complex in Figure 5 in a positive unary basic planar diagram, and then applying lemmas 2.1 and 2.2. The smoothings in this complex have only one strand. Since the rotation number of a smoothing with a unique strand is always 0, we have that the degree shift and the homological degree multiplied by two are in a single diagonal, i.e., $2r - q_r$ is a constant.

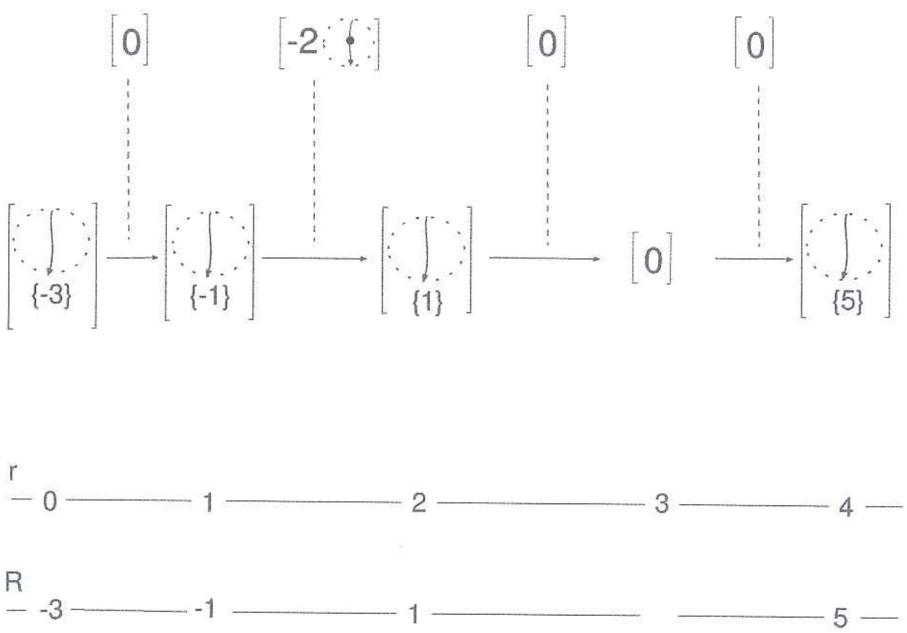


Figure 9. A diagonal complex with only one strand in each of its smoothings.

Corollary 7.4. *(Lee's theorem)The Khovanov complex $Kh(L)$ of a non-split alternating Link L is homotopy equivalent to a complex:*

$$\cdots \longrightarrow \begin{pmatrix} \Phi^r\{q_r+1\} \\ \Phi^r\{q_r-1\} \end{pmatrix} \longrightarrow \begin{pmatrix} \Phi^{r+1}\{q_{r+1}+1\} \\ \Phi^{r+1}\{q_{r+1}-1\} \end{pmatrix} \longrightarrow \cdots$$

where every Φ^r is a matrix of empty 1-manifolds, $q_r = 2r + K$, K a constant, and every differential is a matrix in the ground ring.

Proof. Every non-split alternating link L is obtained by putting a 1-strand tangle T in a 1-input planar diagram with no boundary. Hence, by applying the operator defined from this 1-input planar diagrams to the Khovanov complex of this 1-strand tangle, we obtain the Khovanov complex of a link L. By doing that, the vectors of open arcs that we have in corollary 7.3 become vectors of circles. Moreover, every cobordism of the complex transforms in a multiple of a dotted cylinder. Thus, using Lemma 2.1 converts every single loop in a pair of empty sets $\emptyset\{r + K + 1\}$, $\emptyset\{r + K - 1\}$ and every dotted cylinder in an element of the ground ring. $\square$

Figure 10 displays the closure of the complex in Figure 9. After applying lemmas 2.1 and 2.2 we obtain the complex supported in two lines displayed on Figure 10, as stated in Lee's Theorem.

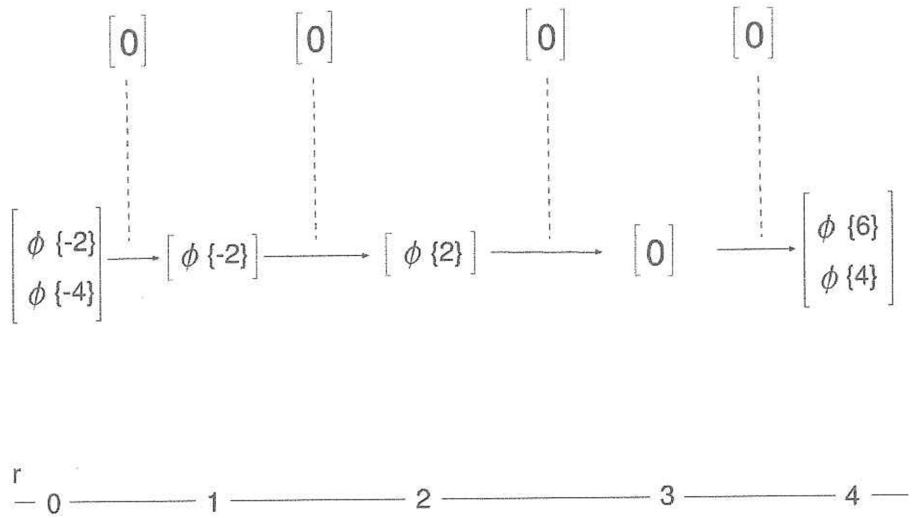


Figure 10. A closure of a coherently diagonal complex is a width-two complex.

Remark 7.5. It is clear that if our ground ring is $\mathbb{Q}$, as in the case of [Lee1], we can use repeatedly lemma 2.2 in the complex in corollary 7.4 and obtain from this complex, one whose differentials are zero, i.e, the Khovanov homology of the link.

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