Toronto Handout on March 3

March-03-13
11:53 AM

Trees and Wheels and Balloons and Hoops

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ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Toronto-1303

15 Minutes on Algebra

Let T be a finite set of "tail labels" and H a finite set of "head labels". Set

$$M_{1/2}(T;H) := FL(T)^H,$$

"H-labeled lists of elements of the degree-completed free Lie algebra generated by T".

$$FL(T) = \left\{ 2t_2 - \frac{1}{2}[t_1,[t_1,t_2]] + \dots \right\} \Big/ \begin{pmatrix}\text{anti-symmetry}\\ \text{Jacobi}\end{pmatrix}$$

... with the obvious bracket.

$$M_{1/2}(u,v;x,y) = \left\{ \left(x \to \text{[tree } u,v \to x],\ y \to -\tfrac{22}{7}\,\text{[trees } v \to y;\ u,v \to y] \right) \dots \right\}$$

Operations $M_{1/2} \to M_{1/2}$ aim! newspeak!

Tail Multiply tm^{uv}_w is $M_{1/2} \to M_{1/2}$ by $\lambda \mapsto \lambda /\!/ (u,v \to w)$, satisfies "meta-associativity", $tm^{uv}_u /\!/ tm^{uw}_u = tm^{vw}_v /\!/ tm^{uv}_u$.

Head Multiply hm^{xy}_z is $M_{1/2} \to M_{1/2}$ by $\lambda \mapsto (\lambda \backslash \{x,y\}) \cup (z \to \operatorname{bch}(\lambda_x, \lambda_y))$, where

$$\operatorname{bch}(\alpha,\beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha,\beta]}{2} + \frac{[\alpha,[\alpha,\beta]]+[[\alpha,\beta],\beta]}{12} + \dots$$

satisfies $\operatorname{bch}(\operatorname{bch}(\alpha,\beta),\gamma) = \log(e^x e^y e^z) = \operatorname{bch}(\alpha, \operatorname{bch}(\beta,\gamma))$ and hence meta-associativity, $hm^{xy}_x /\!/ hm^{xz}_x = hm^{yz}_y /\!/ hm^{xy}_x$.

Tail by Head Action tha^{ux} is $M_{1/2} \to M_{1/2}$ by $\lambda \mapsto \lambda /\!/ RC^{\lambda_x}_u$, where $C^{-\gamma}_u \colon FL \to FL$ is the substitution $u \to e^{-\gamma} u e^{\gamma}$, or more precisely,

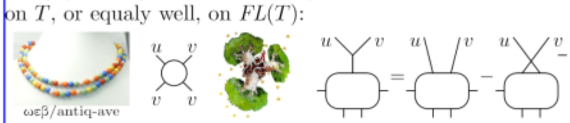
$$C^{-\gamma}_u \colon u \to e^{-\operatorname{ad}\gamma}(u) = u - [\gamma,u] + \frac{1}{2}[\gamma,[\gamma,u]] - \dots,$$

and RC^γ_u is the inverse of that. Note that $C^{\operatorname{bch}(\alpha,\beta)}_u = C^{\alpha /\!/ RC^{-\beta}_u}_u /\!/ C^\beta_u$ and hence "meta $u^{xy} = (u^x)^y$",

$$hm^{xy}_z /\!/ tha^{uz} = tha^{ux} /\!/ tha^{uy} /\!/ hm^{xy}_z,$$

and $tm^{uv}_w /\!/ C^{\gamma /\!/ tm^{uv}_w}_w = C^{\gamma /\!/ RC^{-\gamma}_v}_u /\!/ C^\gamma_v /\!/ tm^{uv}_w$ and hence "meta $(uv)^x = u^x v^x$", $tm^{uv}_w /\!/ tha^{wx} = tha^{ux} /\!/ tha^{vx} /\!/ tm^{uv}_w$.

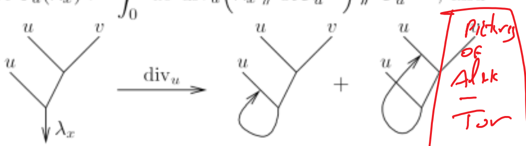
Wheels. Let $M(T;H) := M_{1/2}(T;H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T, or equaly well, on $FL(T)$:



Operations. On $M(T;H)$, define tm^{uv}_w and hm^{xy}_z as before, and tha^{ux} by adding some J-spice:

$$(\lambda;\omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) /\!/ RC^{\lambda_x}_u,$$

where $J_u(\lambda_x) := \int_0^1 ds\, \operatorname{div}_u\left(\lambda_x /\!/ RC^{s\lambda_x}_u\right) /\!/ C^{-s\lambda_x}_u$, and

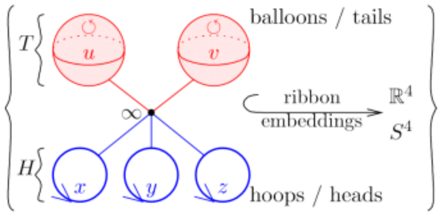


Theorem Green. All green identities still hold.

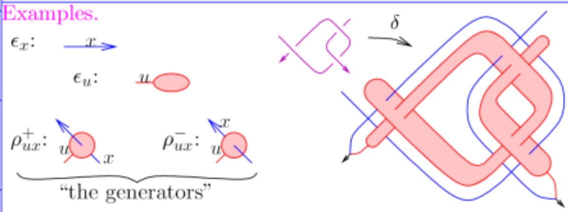
Merge Operation. $(\lambda_1;\omega_1) * (\lambda_2;\omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$.

15 Minutes on Topology

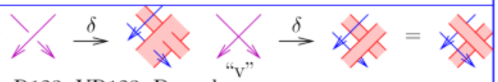
$\mathcal{K}^{bh}(T;H)$.



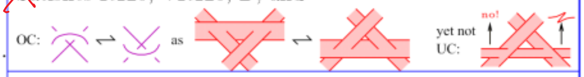
Examples.



More on δ



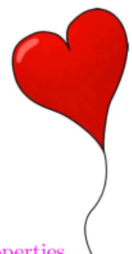
Satisfies R123, VR123, D, and

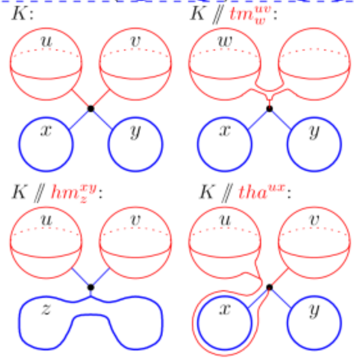


- δ injects u-Knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- δ maps v-tangles to $\mathcal{K}^{bh}$; the kernel is as above, and conjecturally, that's all. Allowing punctures and cuts, δ is onto.

Operations Punctures & Cuts | Connected Sums.

If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

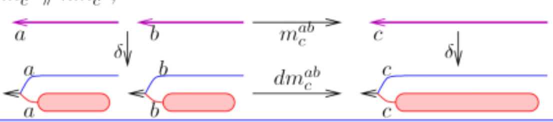




Properties.

- Associativities: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$, for $m = tm, hm$.
- Action axiom t: $tm^{uv}_w /\!/ tha^{wx} = tha^{ux} /\!/ tha^{vx} /\!/ tm^{uv}_w$.
- Action axiom h: $hm^{xy}_z /\!/ tha^{uz} = tha^{ux} /\!/ tha^{uy} /\!/ hm^{xy}_z$.

Tangle concatenations $\to$ $\pi_1 \ltimes \pi_2$. With $dm^{ab}_c := tha^{ab} /\!/ tm^{ab}_c /\!/ hm^{ab}_c$,

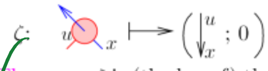


Moral. To construct an M-valued invariant ζ of (v-)tangles, and nearly an invariant on $\mathcal{K}^{bh}$, it is enough to declare ζ on the generators, and verify the relations that δ satisfies.

It is the universal solution to a topological problem and it has many siblings (who talk to each other). It is explicitly computable. Its target space is in itself a space of "universal formulas in Lie algebras" (that's "the miracle"). It seems to be a complete(?) evaluation a certain gauge theory. It is related to a deep conjecture in Lie theory proven by Alekseev and Meinrenken. It has even-better-computable specializations, including one which is an "ultimate Alexander invariant". And plenty of work remains to be done.

Trees and Wheels and Balloons and Hoops: Why I Care

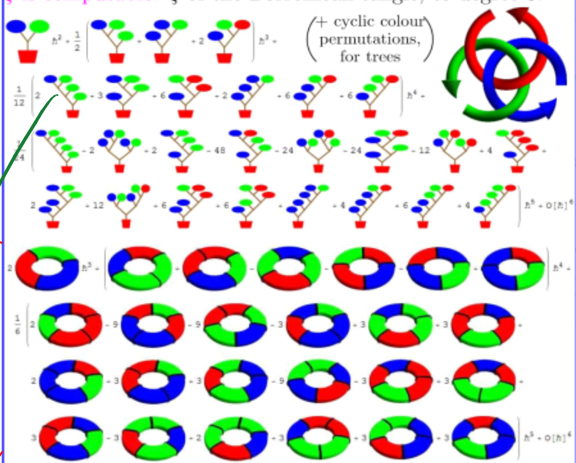
The Invariant ζ. Set $\zeta(\epsilon_x) = (x \to 0; 0)$, $\zeta(\epsilon_u) = ((); 0)$, and



Theorem. ζ is (the log of) the unique homomorphic universal finite type invariant.

... and it is the tip of an iceberg ...

ζ is computable! ζ of the Borromean tangle, to degree 5:



Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T;H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T;H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\mathrm{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be the integral of B (transported via A to ∞) on γ_u.


Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T;H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B)} \bigotimes_x \mathrm{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

Issues. How exactly is B transported via A to ∞? How does the ribbon condition arise? Or if it doesn't, could it be that ζ can be generalized??

The β quotient, 1. • Arises when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra.
• Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Paper in progress: ωεβ/kbh **Class next year:** ωεβ/1350

The β quotient. Let $R = \mathbb{Q}[\![\{c_u\}_{u \in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to (\bar\lambda; \omega) \quad \text{with} \quad \bar\lambda = \sum_{u \in T} \lambda_{ux} ux, \quad \lambda_{ux}, \omega \in R,$$

$$\mathrm{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u + c_v} - 1}\left(\frac{e^{c_u}-1}{c_u} u + e^{c_u}\frac{e^{c_v}-1}{c_v} v\right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u /\!/ RC_u^\lambda = \left(1 + c_u \lambda_u \frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1}\left(e^{c_\lambda} u - c_u \frac{e^{c_\lambda}-1}{c_\lambda} \sum_{v \neq u} \lambda_v v\right),$$

$\mathrm{div}_u \lambda = c_u \lambda_u$, and the ODE for J integrates to

$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda} c_u \lambda_u\right),$$

so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x : \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v \lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u \lambda_{ux} \frac{e^{c_x}-1}{c_x}$ and $\omega \to \log \omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get "β calculus".

β Calculus. Let $\beta(H,T)$ be

$$\left\{\begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l} \omega \text{ and the } \alpha_{ux}\text{'s are} \\ \text{rational functions in} \\ \text{variables } t_u, \text{ one for} \\ \text{each } u \in T. \end{array}\right\},$$


Selmani & B-N, ωεβ/meta

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha + \beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha + \beta + \langle\alpha\rangle\beta & \gamma \end{array},$$

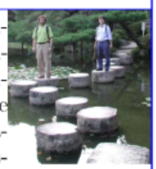
$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v \neq u} \gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why bother? An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination!). *If there should be an Alexander invariant with an algebraic categorifcation, it is this one!*



See also ωεβ/regina, ωεβ/gwu.

Why bother? (2) Related to A-T, K-V, and E-K, should have vast generalization beyond w-knots and the Alexander polynomial.

See also ωεβ/wko, ωεβ/caen, ωεβ/swiss

Handouts to insert:

Tennessee, SwissKnots/Strasbourg, Bonn