Trees and Wheels and Balloons and Hoops

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ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Toronto-1303

15 Minutes on Algebra

Let T be a finite set of "tail labels" and H a finite set of "head labels". Set

$$M_{1/2}(T;H) := FL(T)^H,$$

"H-labeled lists of elements of the degree-completed free Lie algebra generated by T".

$$FL(T) = \left\{2t_2 - \frac{1}{2}[t_1,[t_1,t_2]] + \dots\right\} \Big/ \left(\begin{matrix}\text{anti-symmetry}\\ \text{Jacobi}\end{matrix}\right)$$

... with the obvious bracket.

$$M_{1/2}(u,v;x,y) = \left\{\left(x \to \text{[tree } u,v \to x\text{]},\ y \to \text{[}v \to y\text{]} - \frac{22}{7}\text{[tree } u,v \to y\text{]}\right)\dots\right\}$$

Operations

Tail Multiply tm^{uv}_w: $M_{1/2} \to M_{1/2}$ by $\lambda \mapsto \lambda \,/\!/\, (u, v \to w)$, satisfies "meta-associativity", $tm^{uv}_u \,/\!/\, tm^{uw}_u = tm^{vw}_v \,/\!/\, tm^{uv}_u$.

Head Multiply hm^{xy}_z: $M_{1/2} \to M_{1/2}$ by $\lambda \mapsto (\lambda\backslash\{x,y\}) \cup (z \to \text{bch}(\lambda_x, \lambda_y))$, where

$$\text{bch}(\alpha,\beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha,\beta]}{2} + \frac{[\alpha,[\alpha,\beta]]+[[\alpha,\beta],\beta]}{12} + \dots$$

satisfies $\text{bch}(\text{bch}(\alpha,\beta),\gamma) = \log(e^x e^y e^z) = \text{bch}(\alpha,\text{bch}(\beta,\gamma))$ and hence meta-associativity, $hm^{xy}_x \,/\!/\, hm^{xz}_x = hm^{yz}_y \,/\!/\, hm^{xy}_x$.

Tail by Head Action tha^{ux}: $M_{1/2} \to M_{1/2}$ by $\lambda \mapsto \lambda \,/\!/\, RC^{\lambda_x}_u$, where $C^{-\gamma}_u$: $FL \to FL$ is the substitution $u \to e^{-\gamma}ue^{\gamma}$, or more precisely,

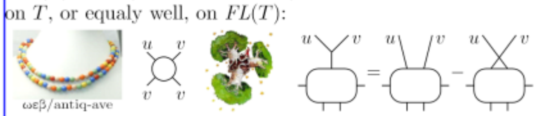
$$C^{-\gamma}_u: u \to e^{-\text{ad}\,\gamma}(u) = u - [\gamma,u] + \frac{1}{2}[\gamma,[\gamma,u]] - \dots,$$

and RC^γ_u is the inverse of that. Note that $C^{\text{bch}(\alpha,\beta)}_u = C^{\alpha /\!/ RC^{-\beta}_u}_u \,/\!/\, C^\beta_u$ and hence "meta $u^{xy} = (u^x)^y$",

$$hm^{xy}_z \,/\!/\, tha^{uz} = tha^{ux} \,/\!/\, tha^{uy} \,/\!/\, hm^{xy}_z,$$

and $tm^{uv}_w \,/\!/\, C^{\gamma}_w = C^{\gamma /\!/ tm^{uv}_w}_u \,/\!/\, C^{\gamma /\!/ RC^{-\gamma}_u}_v \,/\!/\, tm^{uv}_w$ and hence "meta $(uv)^x = u^x v^x$", $tm^{uv}_w \,/\!/\, tha^{wx} = tha^{ux} \,/\!/\, tha^{vx} \,/\!/\, tm^{uv}_w$.

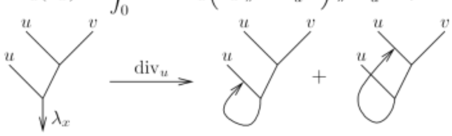
Wheels. Let $M(T;H) := M_{1/2}(T;H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T, or equaly well, on $FL(T)$:



Operations. On $M(T;H)$, define tm^{uv}_w and hm^{xy}_z as before, and tha^{ux} by adding some J-spice:

$$(\lambda;\omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) \,/\!/\, RC^{\lambda_x}_u,$$

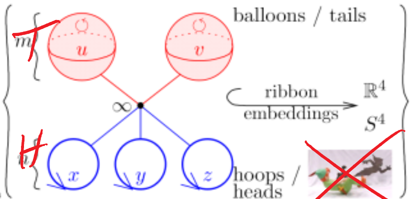
where $J_u(\lambda_x) := \int_0^1 ds\, \text{div}_u\left(\lambda_x \,/\!/\, RC^{s\lambda_x}_u\right) \,/\!/\, C^{-s\lambda_x}_u$, and



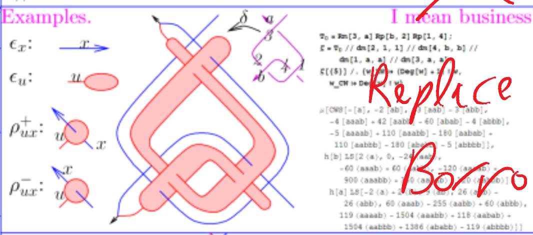
Theorem Blue. All blue identities still hold.

Merge Operation. $(\lambda_1;\omega_1) * (\lambda_2;\omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$.

$\mathcal{K}^{bh}(T;H)$.

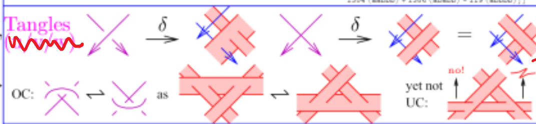


Examples. **I mean business!**



Replace by Borromean

Tangles



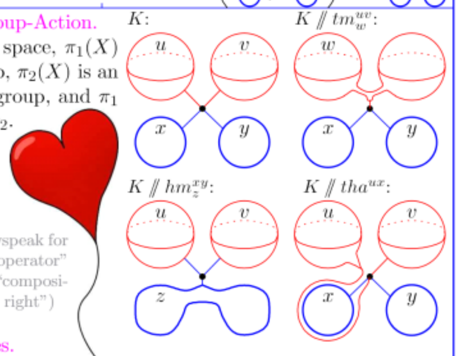
add: satisfies R123, VR123, D

- δ injects u-Knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- δ maps v/w-tangles map to $\mathcal{K}^{bh}$; the kernel contains Reidemeister moves and the "overcrossings commute" relation, and conjecturally, that's all. Allowing punctures and cuts, δ is onto.

Operations Punctures & Cuts. Connected Sums.

Meta-Group-Action. If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

"MGA"



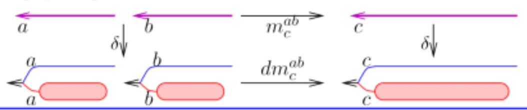
"/\/" is newspeak for "apply an operator" and for "composition left to right")

Properties.

- Associativities: $m^{ab}_a \,/\!/\, m^{ac}_a = m^{bc}_b \,/\!/\, m^{ab}_a$, for $m = tm, hm$.
- Action axiom t: $tm^{uv}_w \,/\!/\, tha^{wx} = tha^{ux} \,/\!/\, tha^{vx} \,/\!/\, tm^{uv}_w$,
- Action axiom h: $hm^{xy}_z \,/\!/\, tha^{uz} = tha^{ux} \,/\!/\, tha^{uy} \,/\!/\, hm^{xy}_z$.

Meta-associativity. $m^{ab}_a \,/\!/\, m^{ac}_a = m^{bc}_b \,/\!/\, m^{ab}_a$.

Tangle concatenations $\to \pi_1 \ltimes \pi_2$. With $dm^{ab}_c := tha^{ab} \,/\!/\, tm^{ab}_c \,/\!/\, hm^{ab}_c$,



Trees and Wheels and Balloons and Hoops and Why I Care

The β quotient, 2. Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u,v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to (\bar\lambda;\omega) \quad \text{with } \bar\lambda = \sum_{x\in H, u\in T} \lambda_{ux} ux, \quad \lambda_{ux}, \omega \in R,$$

$$\text{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v} - 1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

modernize

$$\mu \to (\lambda;\omega) \quad \text{with } \lambda = \sum_{x\in H,\, u\in T} \lambda_{ux} ux, \quad \lambda_{ux}, \omega \in R,$$

$$\mathrm{bch}(u,v) \to \frac{c_u+c_v}{e^{c_u+c_v}-1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u /\!\!/ RC_u^\lambda = \left(1 + c_u\lambda_u \frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1}\left(e^{c_\lambda}u - c_u\frac{e^{c_\lambda}-1}{c_\lambda}\sum_{v\neq u}\lambda_v v\right),$$

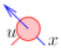
$\mathrm{div}_u\,\lambda = c_u\lambda_u$, and the ODE for J integrates to

$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda}c_u\lambda_u\right),$$

so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x:\lambda_{ux});\omega)$, set $c_x := \sum_v c_v\lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u\lambda_{ux}\frac{e^{c_x}-1}{c_x}$ and $\omega \to \log\omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get "β calculus".

The Invariant ζ. Set $\zeta(\rho^\pm) = (\pm u_x; 0)$. This at least defines an invariant of u/v/w-tangles, and if the topologists will deliver a "Reidemeister" theorem, it is well defined on $\mathcal{K}^{bh}$.

ζ:  $\longmapsto \left(x : +\middle|^{u}\ ; 0\right)$  $\longmapsto \left(x : -\middle|^{u}\ ; 0\right)$

Theorem. ζ is (the log of) a universal finite type invariant (a homomorphic expansion) of w-tangles.

Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \mathrm{Fun}(\oplus_T\mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \mathrm{Fun}(\oplus_T\mathfrak{g})$. Together, $\tau : M(T;H) \to \mathrm{Fun}(\oplus_T\mathfrak{g} \to \oplus_H\mathfrak{g})$, and hence

$$e^\tau : M(T;H) \to \mathrm{Fun}(\oplus_T\mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\mathrm{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be the integral of B (transported via A to ∞) on γ_u.

Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T;H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B\wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B))} \bigotimes_x \mathrm{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

Issues. How exactly is B transported via A to ∞? How does the ribbon condition arise? Or if it doesn't, could it be that ζ can be generalized??

The β quotient, 1. • Arises when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra.

• Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Paper in progress: ωεβ/kbh Class next year: ωεβ/1350

β Calculus. Let $\beta(H,T)$ be

$$\left\{\begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l}\omega \text{ and the } \alpha_{ux}\text{'s are} \\ \text{rational functions in} \\ \text{variables } t_u, \text{ one for} \\ \text{each } u \in T.\end{array}\right\},$$

In preparation, Selmani & B-N.

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

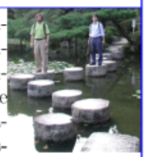
$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why bother? (1) An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination!). *If there should be an Alexander invariant to have an algebraic categorifcation, it is this one!*

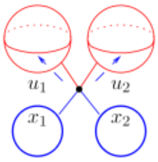


See also ωεβ/regina, ωεβ/gwu.

Why bother? (2) Related to A-T, K-V, and E-K, should have vast generalization beyond w-knots and the Alexander polynomial. See also ωεβ/wko, ωεβ/caen, ωεβ/swiss.

Trees and Wheels and Balloons and Hoops - Extras / Recycling

Invariant #0. With Π_1 denoting "honest π_1", map $\gamma \in \mathcal{K}^{bh}(m,n)$ to the triple $(\Pi_1(\gamma^c),(u_i),(x_j))$, where the meridian of the balls u_i normally generate Π_1, and the "longtitudes" x_j are some elements of Π_1. $*$ acts like $*$, tm acts by "merging" two meridians/generators, hm acts by multiplying two longtitudes, and tha^{ux} acts by "conjugating a meridian by a longitude":



Not computable!
(but nearly)

$$(\Pi,(u,\dots),(x,\dots)) \mapsto (\Pi*\langle\bar u\rangle/(u=x\bar u x^{-1}),(\bar u,\dots),(x,\dots))$$

Failure #0. Can we write the x's as free words in the u's? If $x=uv$, compute $x \mathbin{/\!\!/} tha^{ux}$:

$$x=uv \to \bar u v = u^x v = u^{\bar u v}v = u^{u^x v}v = u^{u^{u^x v}v}v = \cdots$$

Why ODEs? **Q.** Find f s.t. $f(x+y)=f(x)f(y)$. **A.** $\frac{df(s)}{ds} = \frac{d}{d\epsilon}f(s+\epsilon) = \frac{d}{d\epsilon}f(s)f(\epsilon) = f(s)C$. Now solve this ODE using Picard's theorem or power series.

Alekseev Torossian

Scheme. • Balloons and hoops in $\mathbb{R}^4$, algebraic structure and relations with 3D.
• An ansatz for a "homomorphic" invariant: computable, related to finite-type and to BF.
• Reduction to an "ultimate Alexander invariant".

An RC_u^λ example.