# The differentials of exp, bch, C, and RC

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"Non-commutative exponential gymnastics"

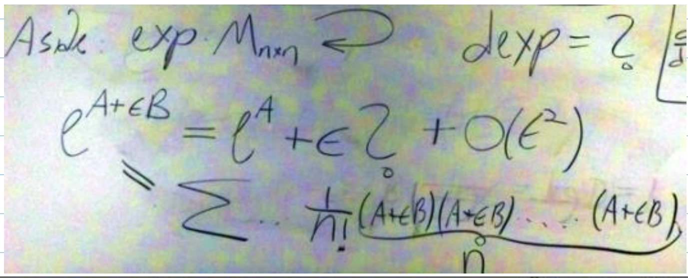
The differential of exp is in http://drorbn.net/drorbn/bbs/show?shot=wClips-120418-132328.jpg and the two shots following it:

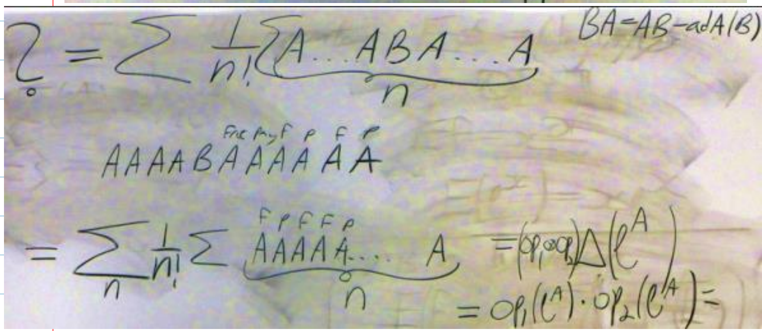
$$\delta e^{\gamma} = e^{\gamma}\left(\frac{1-e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma}\right)(\delta\gamma)$$

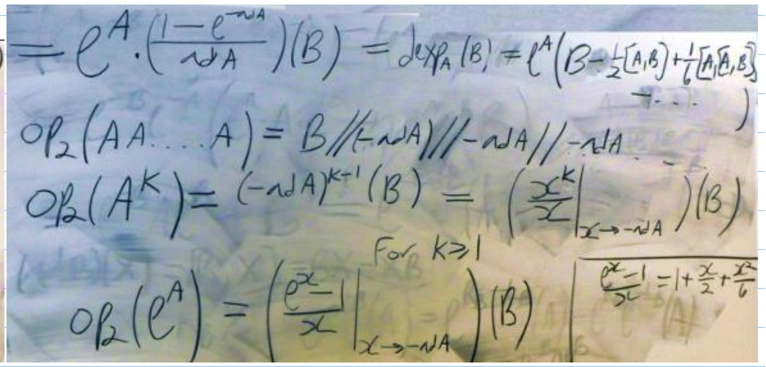
$$\delta\,\operatorname{bch}(\alpha,\beta) =$$

$$\delta C_u^{\gamma} =$$

$$\delta RC_u^{\gamma} =$$




Also, $\delta e^{\gamma} = e^{\gamma}\left(\frac{1-e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma}\right)(\delta\gamma) = \left(e^{\operatorname{ad}\gamma}\left(\frac{1-e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma}\right)(\delta\gamma)\right)e^{\gamma}$

$$= \left(\frac{e^{\operatorname{ad}\gamma}-1}{\operatorname{ad}\gamma}\right)(\delta\gamma)\cdot e^{\gamma}$$

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Suppose $\gamma = \operatorname{bch}(\alpha,\beta)$, so $e^{\gamma} = e^{\alpha}e^{\beta}$. Then

$$e^{\gamma}\left(\frac{1-e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma}\right)(\delta\gamma) = e^{\alpha}\left(\frac{1-e^{-\operatorname{ad}\alpha}}{\operatorname{ad}\alpha}\right)(\delta\alpha)\cdot e^{\beta} + e^{\alpha}e^{\beta}\left(\frac{1-e^{-\operatorname{ad}\beta}}{\operatorname{ad}\beta}\right)(\delta\beta)$$

$$= e^{\alpha}e^{\beta}\left[\left(e^{-\operatorname{ad}\beta}\frac{1-e^{-\operatorname{ad}\alpha}}{\operatorname{ad}\alpha}\right)(\delta\alpha) + \left(\frac{1-e^{-\operatorname{ad}\beta}}{\operatorname{ad}\beta}\right)(\delta\beta)\right]$$

So

$$\left(\frac{1-e^{-\operatorname{ad}\gamma}}{\operatorname{ad}\gamma}\right)(\delta\gamma) = \left(e^{-\operatorname{ad}\beta}\frac{1-e^{-\operatorname{ad}\alpha}}{\operatorname{ad}\alpha}\right)(\delta\alpha) + \left(\frac{1-e^{-\operatorname{ad}\beta}}{\operatorname{ad}\beta}\right)(\delta\beta) \quad \text{or}$$

$$\delta\operatorname{bch}(\alpha,\beta) = \frac{\operatorname{ad}\gamma}{1-e^{-\operatorname{ad}\gamma}}\left(\left(e^{-\operatorname{ad}\beta}\frac{1-e^{-\operatorname{ad}\alpha}}{\operatorname{ad}\alpha}\right)(\delta\alpha) + \left(\frac{1-e^{-\operatorname{ad}\beta}}{\operatorname{ad}\beta}\right)(\delta\beta)\right)$$

$$\delta BCH(\alpha,\beta) = \frac{\quad}{1-e^{-ad\gamma}}\left(\left(\frac{\quad}{ad\alpha} \qquad \frac{\quad}{ad\beta}\right)(\delta\beta)\right)$$

Some play: $e^{-ad\beta} - e^{-ad\gamma} = (1-e^{-ad\gamma}) - (1-e^{-ad\beta})$

So $rhs \simeq \frac{ad\gamma}{ad\gamma}\left(1 - \frac{1-e^{-ad\beta}}{1-e^{-ad\alpha}}\right)(\delta\alpha) + (\ldots)\delta\beta$ not worth the bother.

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$$C_u^\gamma(u) = e^{ad\gamma}(u)$$

$$\delta e^{ad\gamma} = \left(\frac{e^{ad_{ad_\gamma}}-1}{ad_{ad_\gamma}}\right)(ad_{\delta\gamma}) \cdot e^{ad_\gamma}$$

$$\boxed{= ad_{\left(\frac{e^{ad_\gamma}-1}{ad_\gamma}\right)(\delta\gamma)} \cdot e^{ad_\gamma}}$$

$$\left.= e^{ad_\gamma} \cdot ad_{\left(\frac{1-e^{-ad_\gamma}}{ad_\gamma}\right)(\delta\gamma)}\right\} \text{irrelevant}$$

Aside: $[ad\alpha, ad\beta](\gamma) = [\alpha,[\beta,\gamma]] - [\beta,[\alpha,\gamma]]$

$= [[\alpha,\beta],\gamma]$

$= ad[\alpha,\beta](\gamma)$ so

$ad_{ad\alpha}(ad_\beta) = ad_{[\alpha,\beta]}$

$= ad_{ad_\alpha\beta}$

So $\delta C_u^\gamma =$ mess because conjugation automorphisms don't compose well.

$$\sim \left(ad_{\left(\frac{e^{ad_\gamma}-1}{ad_\gamma}\right)(\delta\gamma)} // RC_u^{-\gamma}\right) // C_u^\gamma$$

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$I = C_u^\gamma // RC_u^{-\gamma}$ so, taking $\frac{\delta}{\delta\gamma}$,

$$0 = ad_u\left\{\left(\frac{e^{ad\gamma}-1}{ad\gamma}\right)(\delta\gamma) // RC_u^{-\gamma}\right\} // C_u^\gamma // RC_u^{-\gamma} + C_u^\gamma // \delta RC_u^{-\gamma}$$

Hence

$$\delta RC_u^{-\gamma} = - RC_u^{-\gamma} // ad_u\left\{\left(\frac{e^{ad\gamma}-1}{ad\gamma}\right)(\delta\gamma) // RC_u^{-\gamma}\right\}$$

Hence

$$\delta RC_u^{\gamma} = RC_u^{\gamma} // ad_u\left\{\left(\frac{1-e^{-ad\gamma}}{ad\gamma}\right)(\delta\gamma) // RC_u^{\gamma}\right\}$$

$$\delta RC_u^\gamma = RC_u^\gamma \,/\!/\, ad_u\left\{\left(\frac{1-e^{\cdots}}{ad\gamma}(\delta\gamma)\right) /\!/ RC_u^\gamma\right\}$$