Pensieve header: Trees and Wheels for the Borromean Rings.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2013-03"];
<< FreeLie.m
```

```
Domain[f_List] := First /@ f;
f_ \ key_ := DeleteCases[f, key → _];
f_ \ keys_List := Fold[#1 \ #2 &, f, keys];
f1_List ≡ f2_List := Domain[f1] === Domain[f2] && (And @@ (
       ((# /. f1) ≡ (# /. f2)) & /@ Domain[f1]
     ));
(* LieDerivation[der_][f_List] := MapAt[LieDerivation[der], f, {All, 2}]; *)
LieMorphism[mor_][f_List] := MapAt[LieMorphism[mor], f, {All, 2}];
M /: M[λ1_, ω1_] ⋃ M[λ2_, ω2_] := M[λ1 ⋃ λ2, ω1 + ω2];
M[λ1_, ω1_] ≡ M[λ2_, ω2_] := (λ1 ≡ λ2) && (ω1 ≡ ω2);
```

```
tm[u_, v_, w_][λ_List] := λ // LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩];
tm[u_, v_, w_][M[λ_, ω_]] := LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩] /@ M[λ, ω];
hm[x_, y_, z_][λ_List] := Union[λ \ {x, y}, {z → BCH[x /. λ, y /. λ]}];
hm[x_, y_, z_][M[λ_, ω_]] := M[λ // hm[x, y, z], ω];
RC[u_, λx_LieSeries, ub_][ser_] := StableApply[
   LieMorphism[⟨u⟩ → Ad[λx][⟨ub⟩]],
   ser
  ];
RC[u_, λx_LieSeries][ser_] :=
  ser // RC[⟨u⟩, λx, ⟨"υ"⟩] // LieMorphism[⟨"υ"⟩ → ⟨u⟩];
J[u_, λx_] := Module[{s},
   IntegrateCWSeries[
    div[⟨u⟩, λx // RC[⟨u⟩, s λx]] // LieMorphism[⟨u⟩ → Ad[-s λx][⟨u⟩]],
    {s, 0, 1}
   ]
  ];
tha[u_, x_][λ_List] := MapAt[RC[⟨u⟩, x /. λ], λ, {All, 2}];
tha[u_, x_][M[λ_, ω_]] :=
  M[λ // tha[u, x], (ω + J[⟨u⟩, x /. λ]) // RC[⟨u⟩, x /. λ]];
dm[a_, b_, c_][μ_] := μ // tha[⟨a⟩, b] // tm[⟨a⟩, ⟨b⟩, ⟨c⟩] // hm[a, b, c];
R⁺[u_, x_] := M[{x → MakeLieSeries[⟨u⟩], u → MakeLieSeries[0]}, MakeCWSeries[0]];
R⁻[u_, x_] := M[{x → MakeLieSeries[-⟨u⟩], u → MakeLieSeries[0]}, MakeCWSeries[0]];
```

```
μ0 = R⁻[r, 6] ⋃ R⁺[2, 4] ⋃ R⁻[g, 9] ⋃ R⁺[5, 7] ⋃ R⁻[b, 3] ⋃ R⁺[8, 1]
```

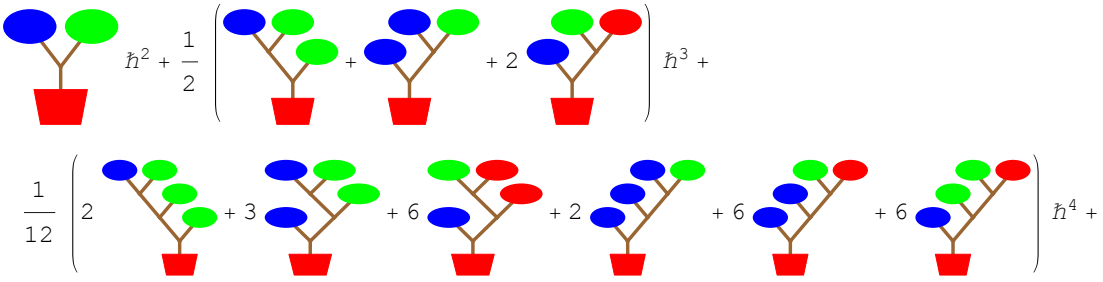
```
M[{1 → LS[8, 0, 0], 2 → LS[0, 0, 0], 3 → LS[-b, 0, 0], 4 → LS[2, 0, 0],
  5 → LS[0, 0, 0], 6 → LS[-r, 0, 0], 7 → LS[5, 0, 0], 8 → LS[0, 0, 0],
  9 → LS[-g, 0, 0], b → LS[0, 0, 0], g → LS[0, 0, 0], r → LS[0, 0, 0]}, CWS[0, 0, 0]]
```

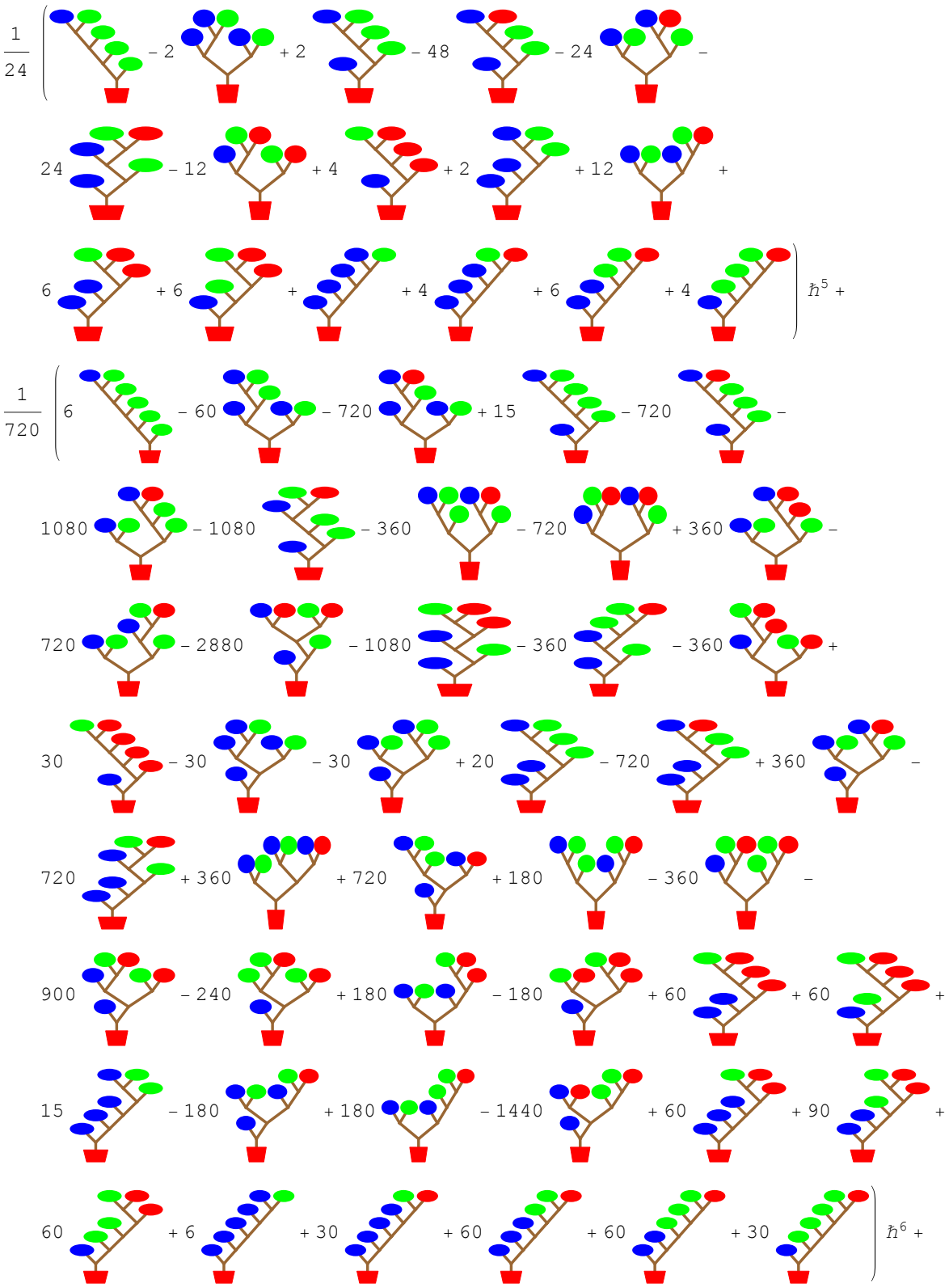
```
Do[μ0 = μ0 // dm[r, k, r], {k, 1, 3}];
Do[μ0 = μ0 // dm[g, k, g], {k, 4, 6}];
Do[μ0 = μ0 // dm[b, k, b], {k, 7, 9}];
μ0
```

$$M\left[\left\{b \to LS\left[0, \overline{gr}, \frac{1}{2}\,\overline{g\,\overline{gr}} + \overline{\overline{br}\,g} + \frac{1}{2}\,\overline{\overline{gr}\,r}\right], g \to LS\left[0, -\overline{br}, \frac{1}{2}\,\overline{b\,\overline{br}} - \overline{b\,\overline{gr}} - \overline{\overline{br}\,g} + \frac{1}{2}\,\overline{\overline{br}\,r}\right],\right.\right.$$
$$\left.\left. r \to LS\left[0, \overline{bg}, \frac{1}{2}\,\overline{b\,\overline{bg}} + \overline{b\,\overline{gr}} + \frac{1}{2}\,\overline{\overline{bg}\,g}\right]\right\}, CWS[0, 0, 2\,CW[bgr]]\right]$$

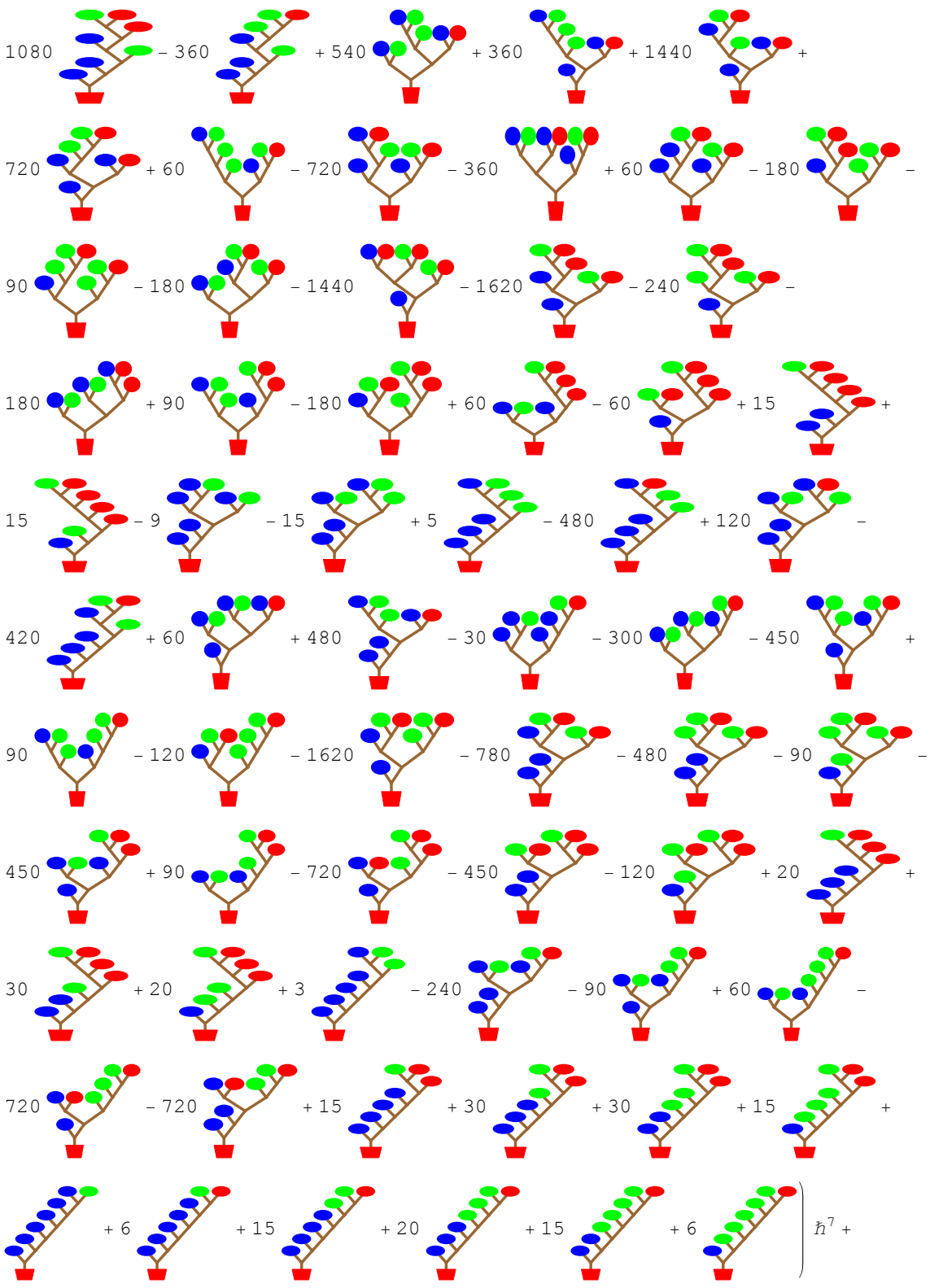
Trees

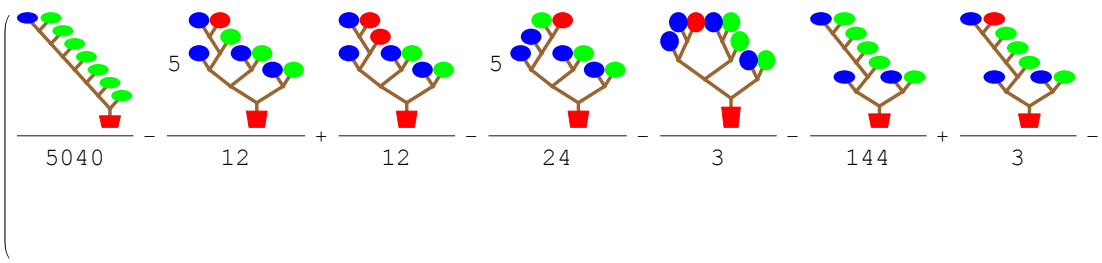
```
trees = Table[(r /. First[μ0])@@k, {k, 8}];
t1 = Series[(List@@trees //. w_LW :> B@@Reverse[LyndonFactorization[w]] /.
       B[s_] :> s /. t_B :> Tree[t]).ℏ^Range[Length[trees]],
   {ℏ, 0, Length[trees]}
  ] /. {"r" → r, "g" → g, "b" → b};
t1 /. t_Tree :> TreeForm[t,
   VertexRenderingFunction → (Switch[#2,
       Tree, {
        Red,
        Polygon[
         {{-0.4, 0.4} - #1, {0.4, 0.4} - #1, {0.3, -0.4} - #1, {-0.3, -0.4} - #1}]
       },
       B, {},
       _, {
        ReleaseHold[#2 /. {r → Red, g → Green, b → Blue}],
        Disk[-#1, 0.4]
       }
      ] &),
   EdgeRenderingFunction → ({
       Brown, Thickness[0.03],
       Line[-#]
      } &),
   PlotRangePadding → 0, ImageSize → 60, AspectRatio → 1
  ]
```

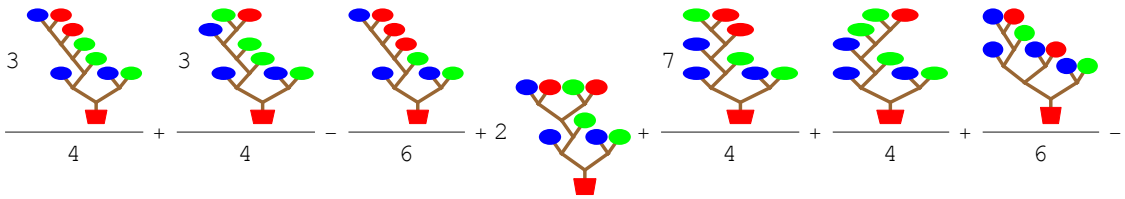


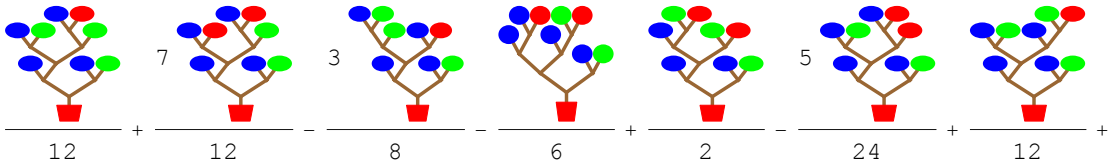


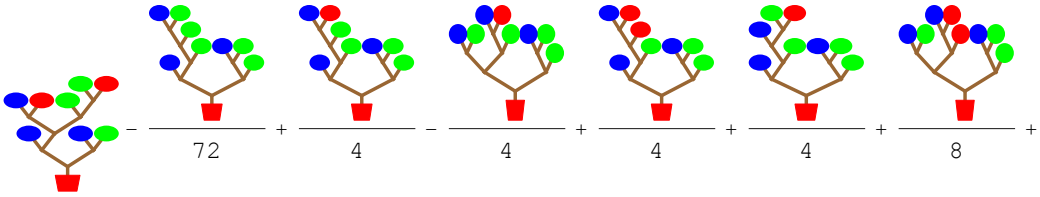
$$\frac{1}{720}\Big[- 120 - 20 + 360 + 360 + 360 + 300 - 15 - 360 - 180 + 3 - 480 - 840 + 360 - 840 - 540 + 360 + 540 - 240 - 1080 + 720 + 540 - 540 - 120 - 360 - 360 + 180 + 360 - 120 - 360 + 720 - 720 - 1080 + 360 - 2160 - 840 - 360 - 1440 - 540 - 120 + 120 - 120 + 90 + 6 + 3 - 30 + 480 + 120 + 240 - 10 + 5 - 720 - 1260 - 360 - 1080 + 420 + 360 - 180 + 360 + 360 - 180 - 1440 - 2160 -$$

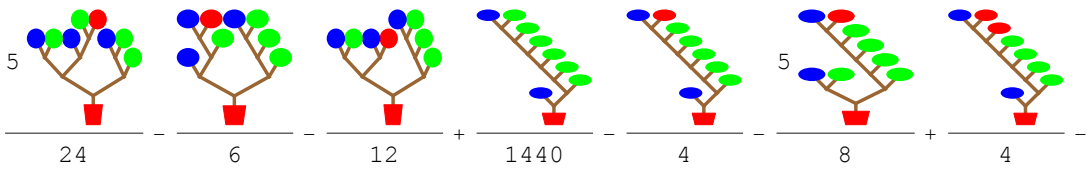

1080 − 360 + 540 + 360 + 1440 +
720 + 60 − 720 − 360 + 60 − 180 −
90 − 180 − 1440 − 1620 − 240 −
180 + 90 − 180 + 60 − 60 + 15 +
15 − 9 − 15 + 5 − 480 + 120 −
420 + 60 + 480 − 30 − 300 − 450 +
90 − 120 − 1620 − 780 − 480 − 90 −
450 + 90 − 720 − 450 − 120 + 20 +
30 + 20 + 3 − 240 − 90 + 60 −
720 − 720 + 15 + 30 + 30 + 15 +
+ 6 + 15 + 20 + 15 + 6 ℏ^7 +

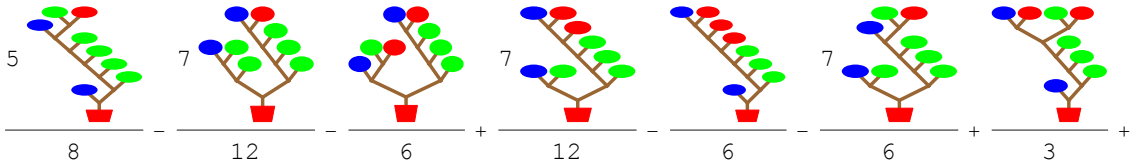


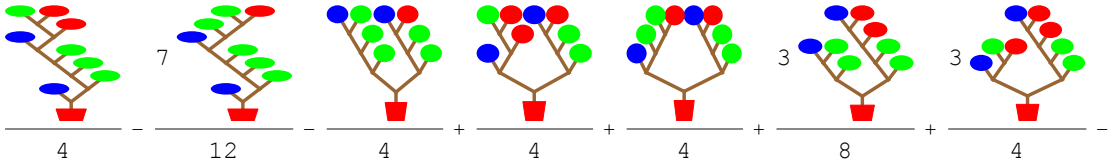


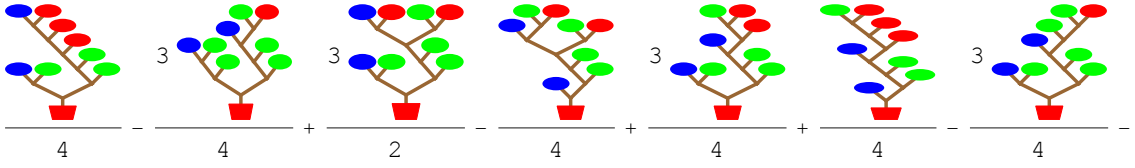


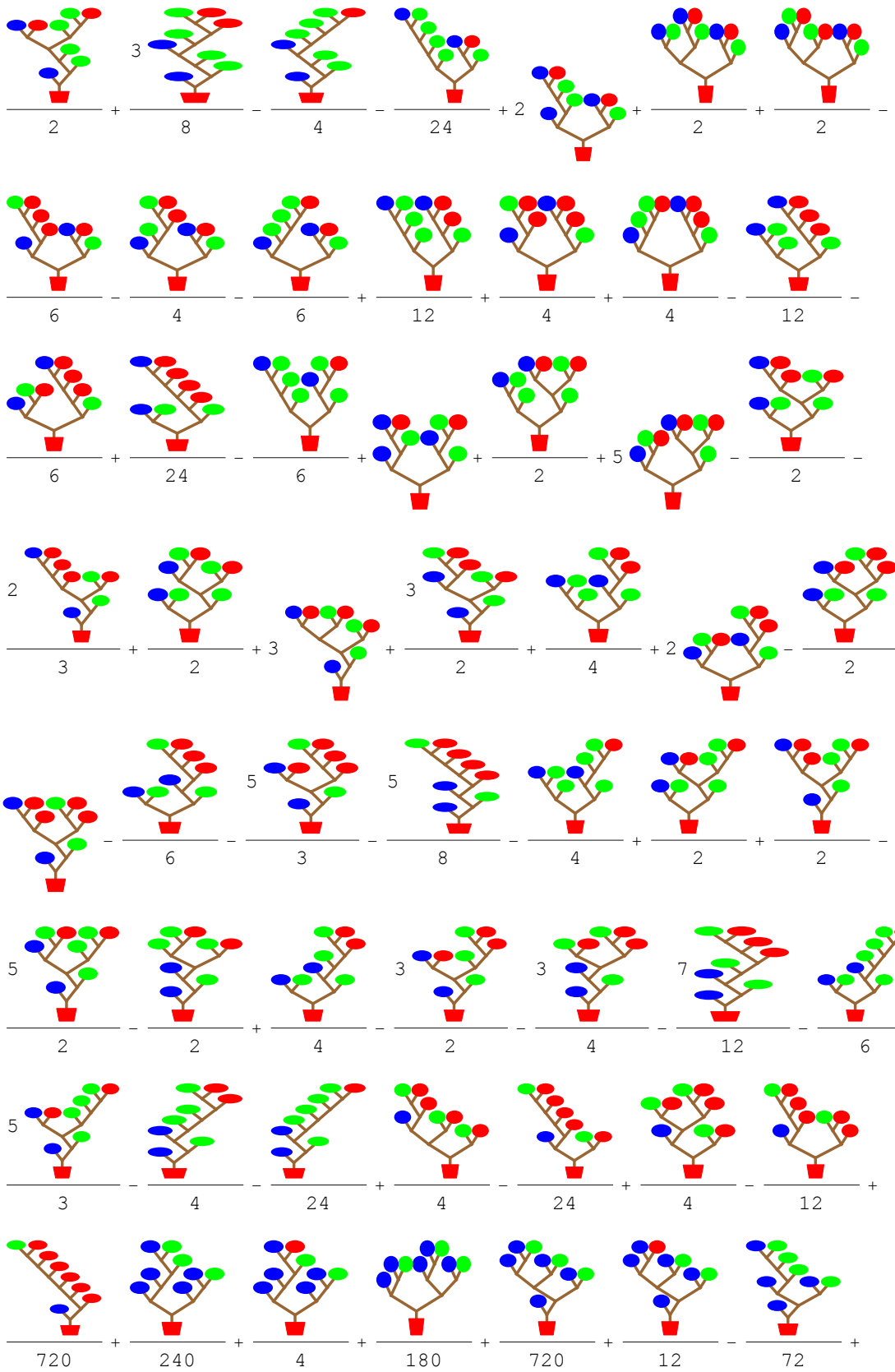


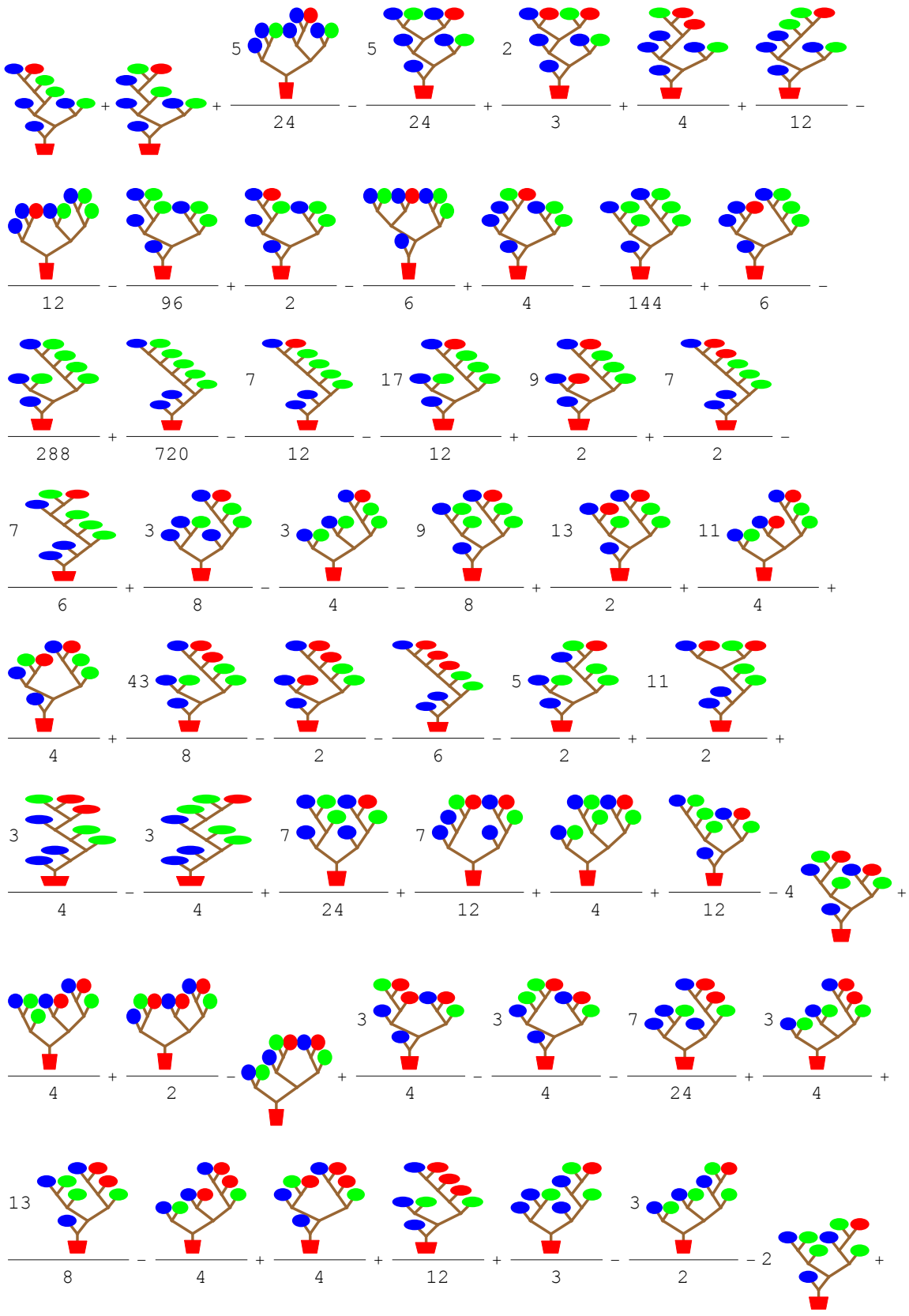


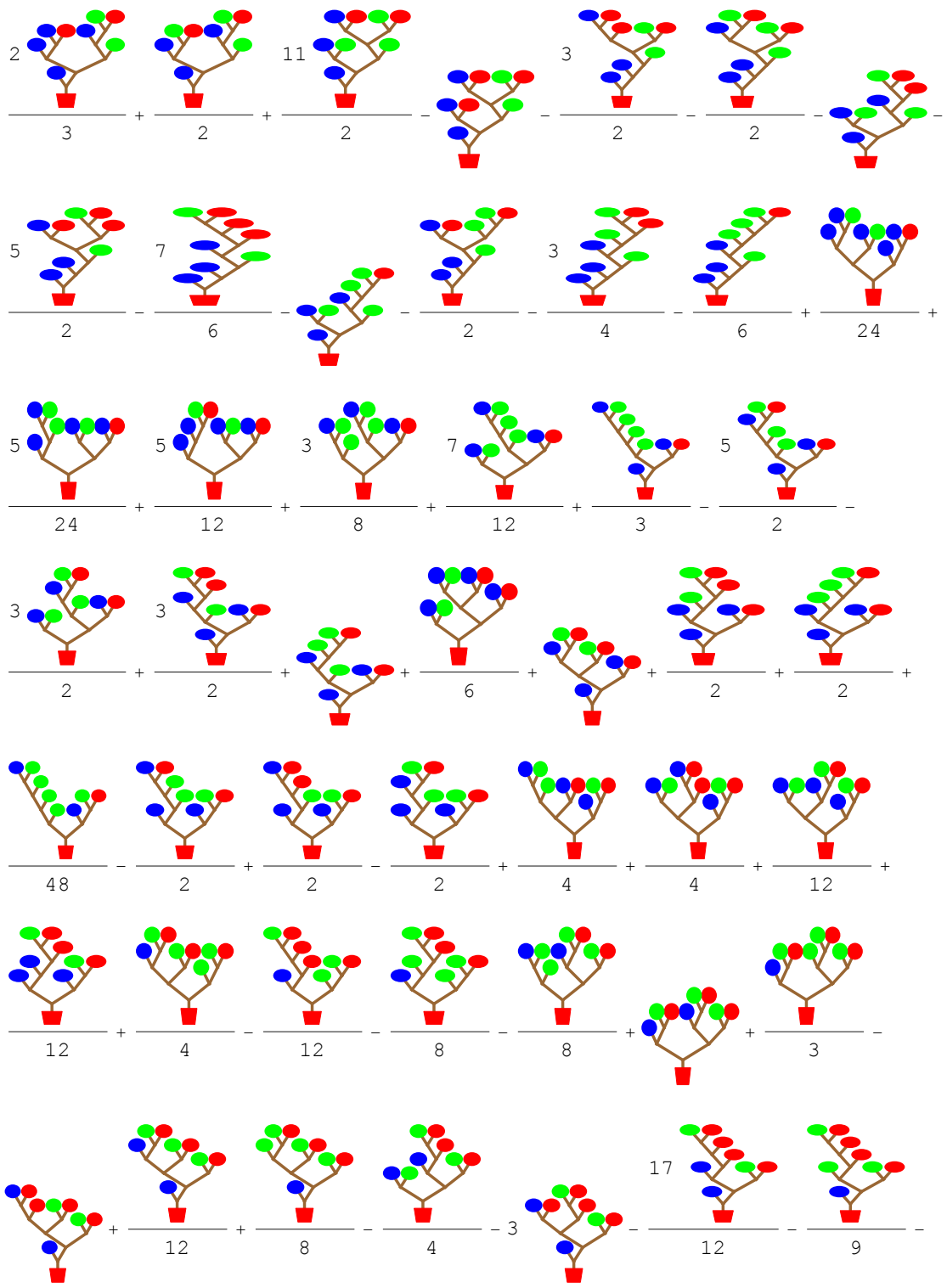


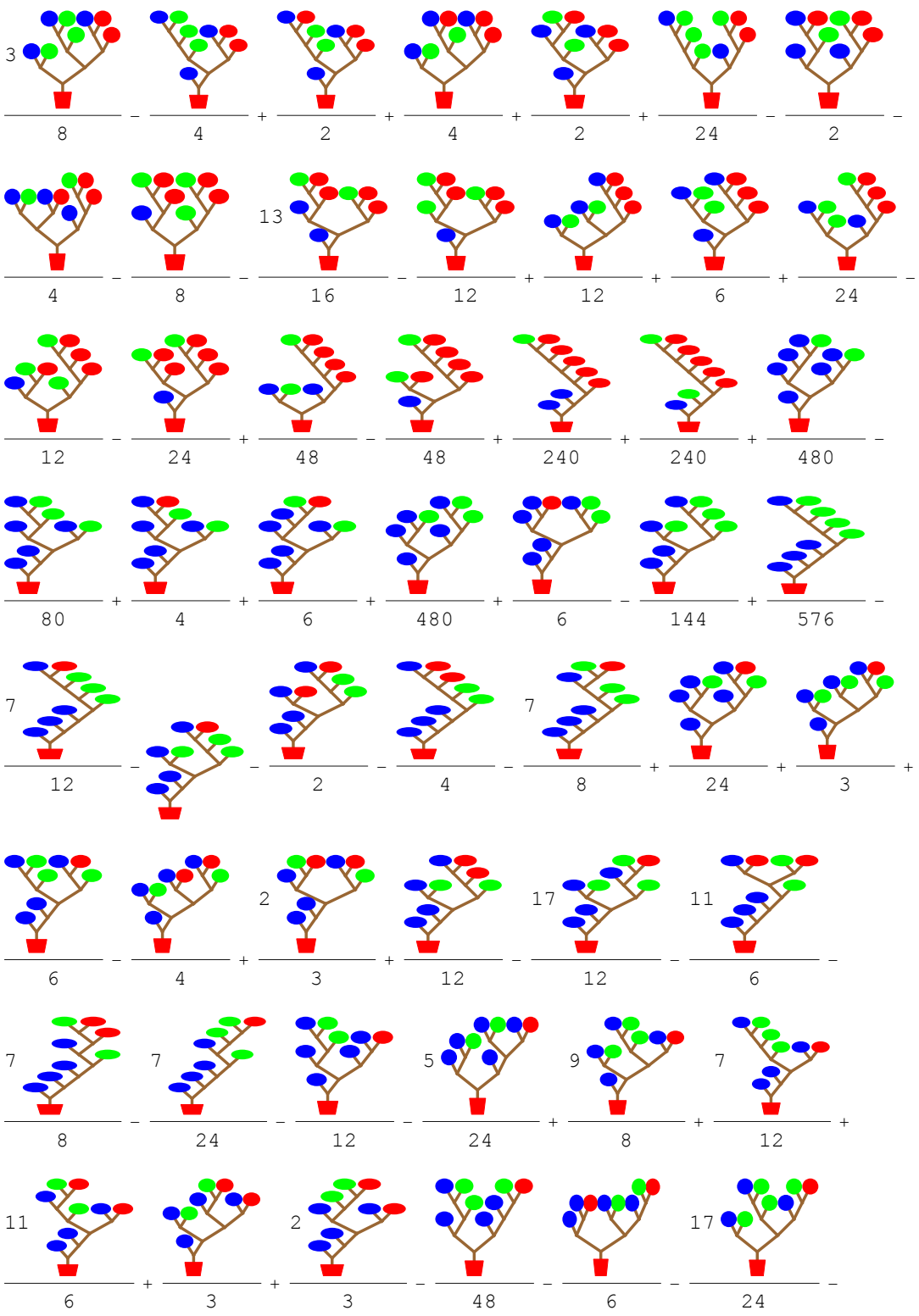


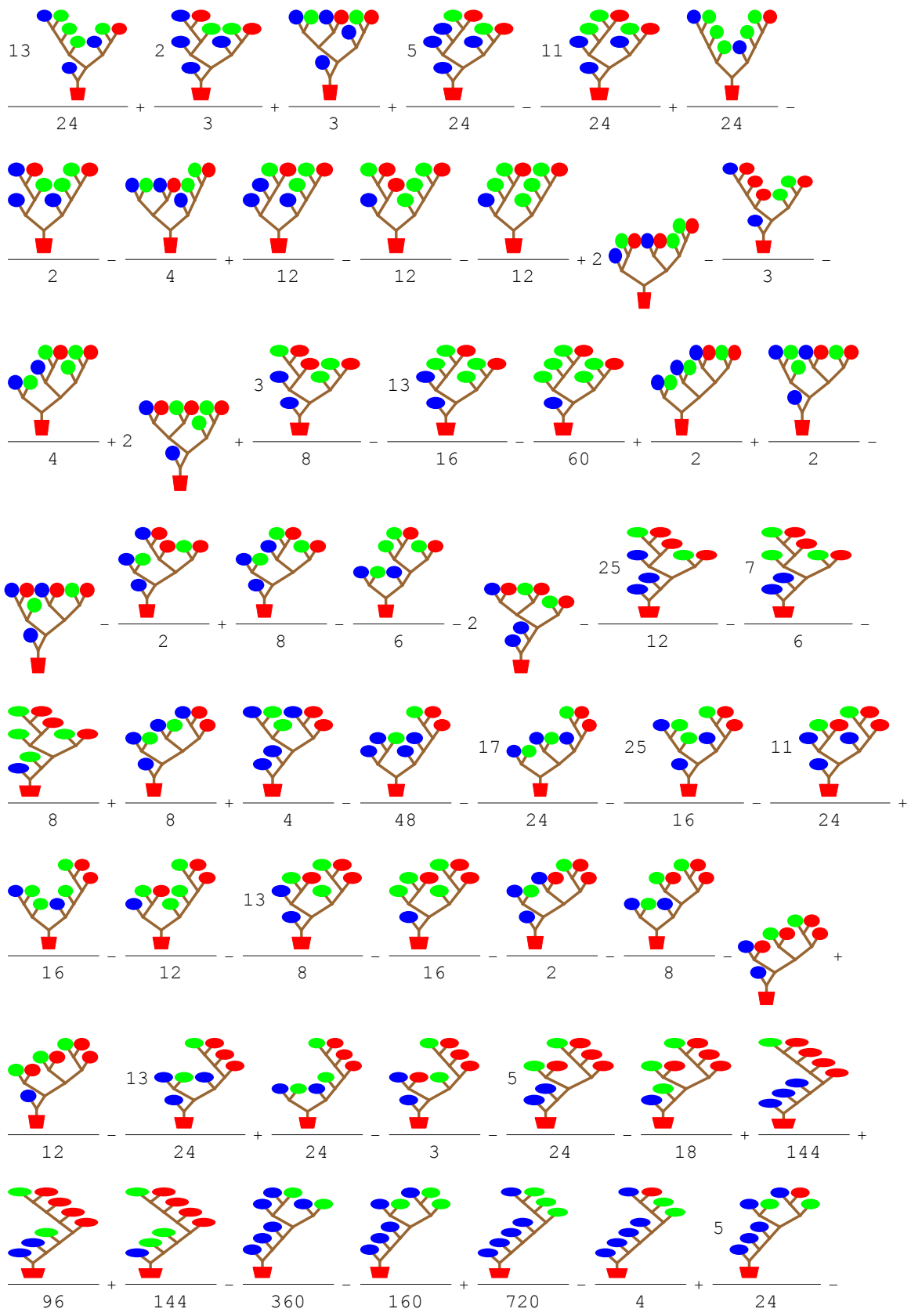


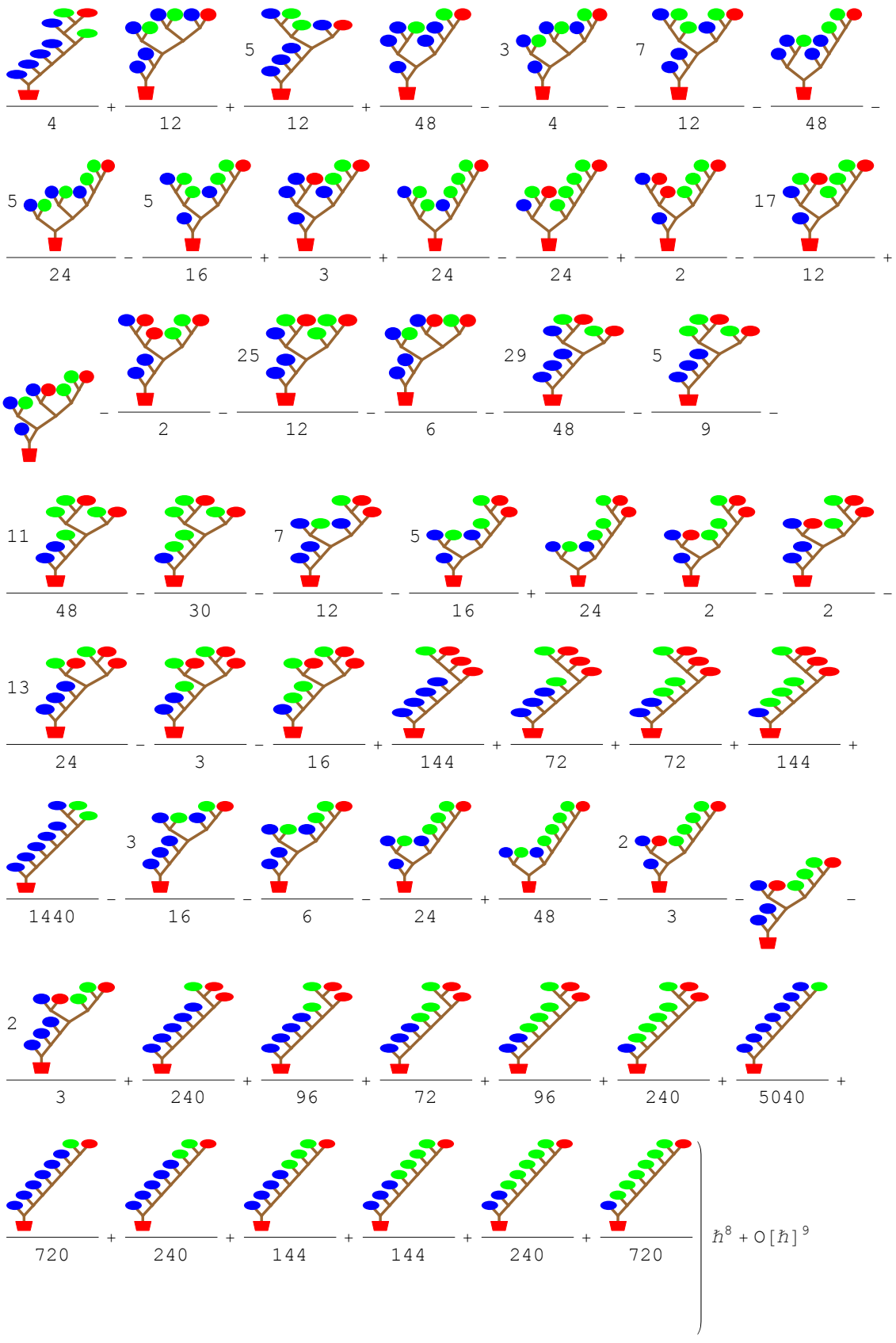






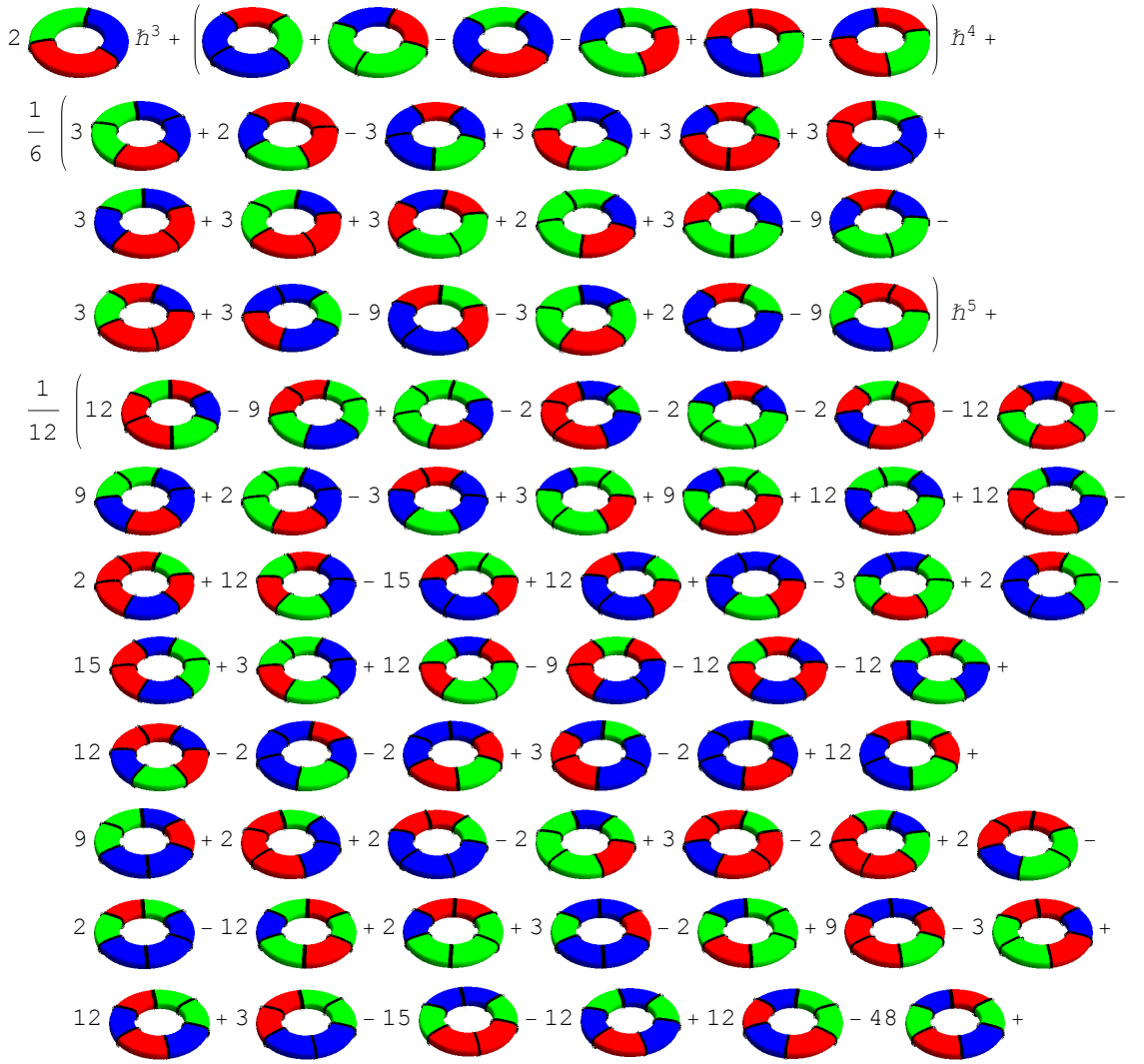


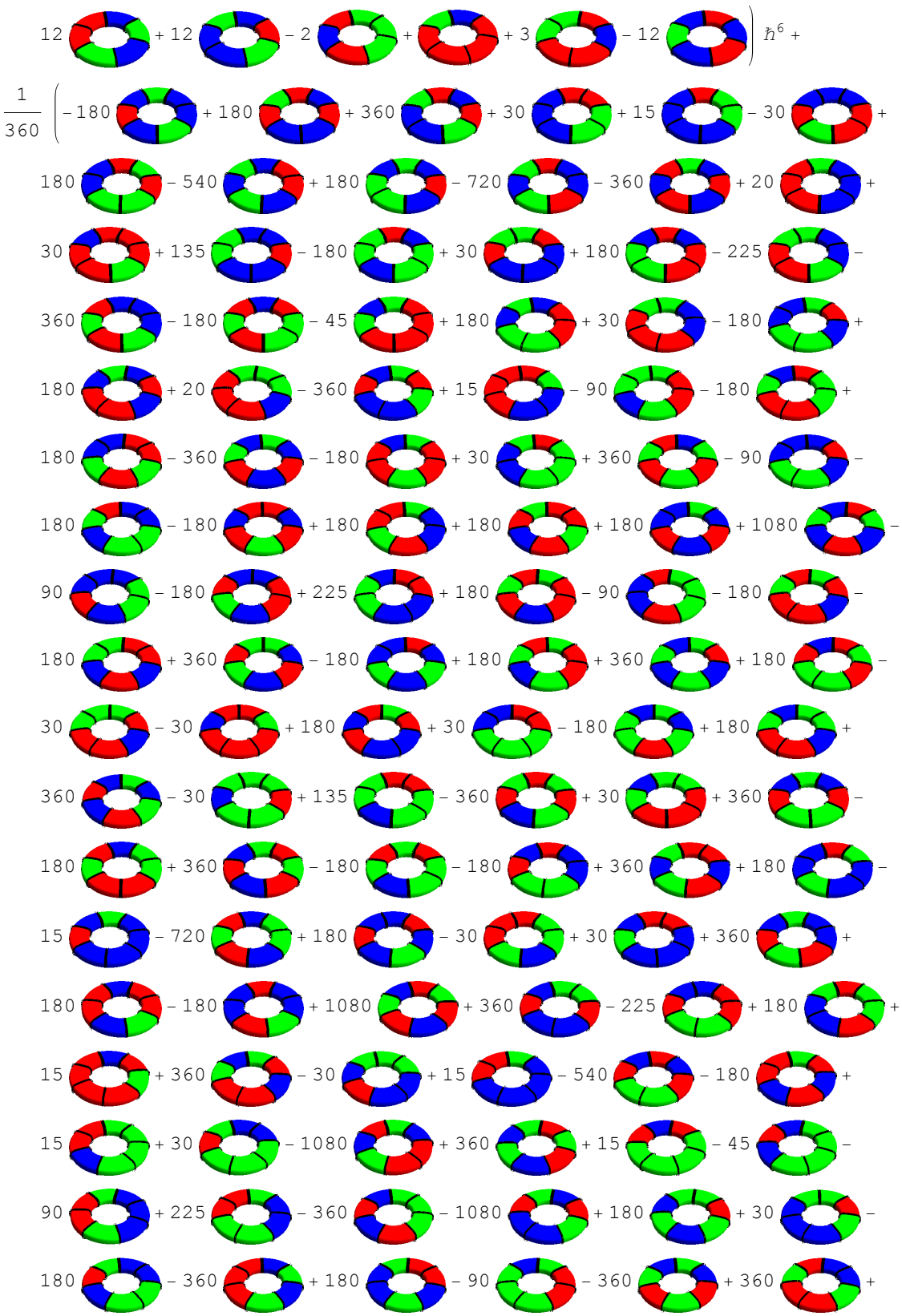


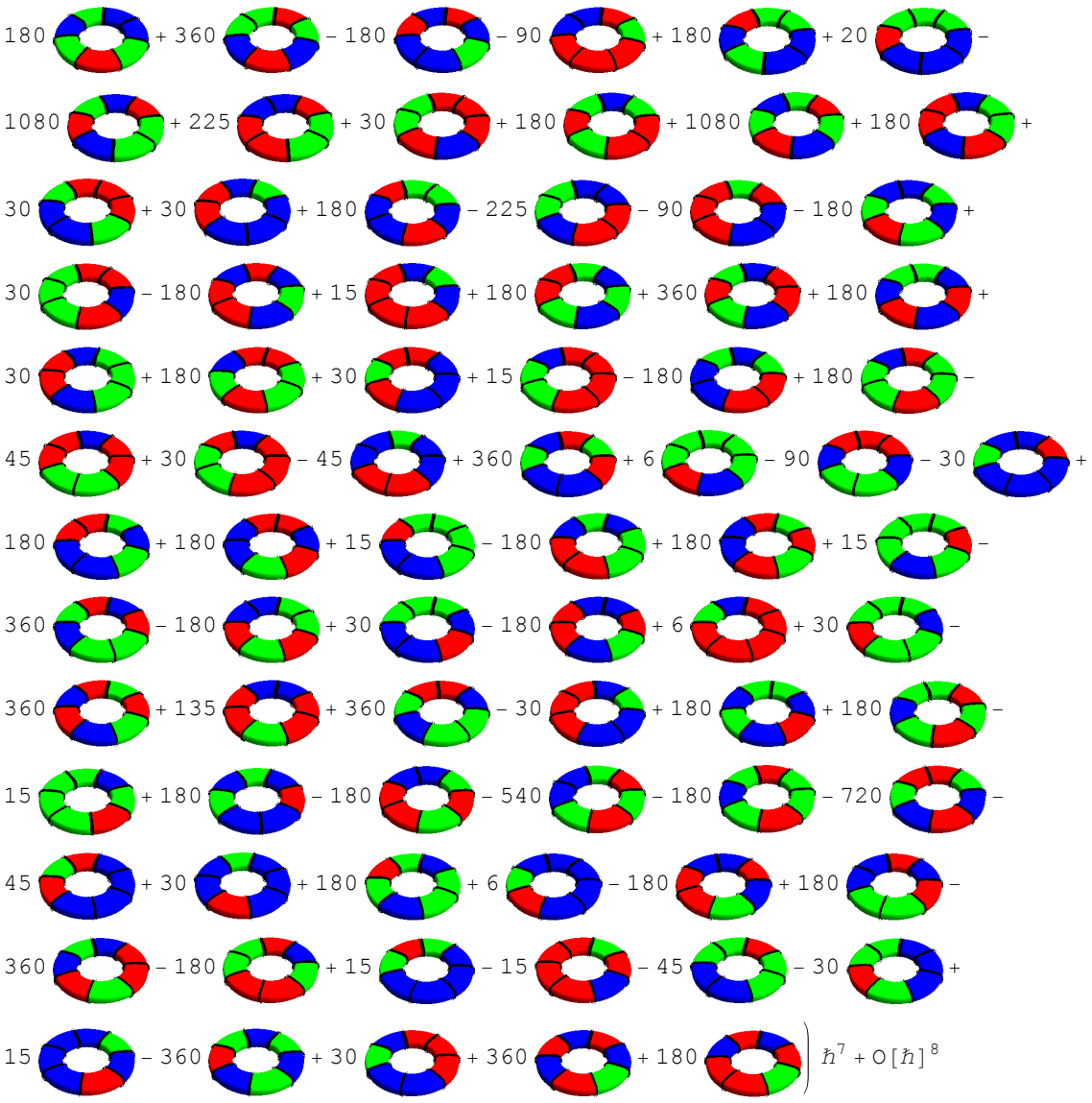

4 + 12 + 5/12 + 48 − 3/4 − 7/12 − 48 −
5/24 − 5/16 + 3 + 24 − 24 + 2 − 17/12 +
2 − 25/12 − 6 − 29/48 − 5/9 −
11/48 − 30 − 7/12 − 5/16 + 24 − 2 − 2 −
13/24 − 3 − 16 + 144 + 72 + 72 + 144 +
1440 − 3/16 − 6 − 24 + 48 − 2/3 −
2/3 + 240 + 96 + 72 + 96 + 240 + 5040 +
720 + 240 + 144 + 144 + 240 + 720
ℏ^8 + O[ℏ]^9

Wheels

```
n = 7;
wheels = Table[Last[μ0]@k, {k, n}];
SetOptions[Rasterize, {RasterSize → 256, ImageSize → 256}];
Collect[
  Expand[(Plus @@ wheels)] /.
   CW[s_String] :> ħ^StringLength[s] Show[ImageCrop[PieChart3D[
        Table[1, {StringLength[s]}],
        ChartStyle :> (Characters[s] /. {"r" → Red, "g" → Green, "b" → Blue}),
        SectorOrigin → {{RandomReal[{0, 2 π}], "Counterclockwise"}, 1},
        ChartBaseStyle → EdgeForm[{Thickness[0.03], Black}],
        ChartElementFunction → "ProfileSector3D",
        ImagePadding → 0, ImageMargins → 0, PlotRangePadding → 0
       ]], ImageSize → 52],
  ħ, Factor] + O[ħ]^(n+1)
```






180 + 360 − 180 − 90 + 180 + 20 −
1080 + 225 + 30 + 180 + 1080 + 180 +
30 + 30 + 180 − 225 − 90 − 180 +
30 − 180 + 15 + 180 + 360 + 180 +
30 + 180 + 30 + 15 − 180 + 180 −
45 + 30 − 45 + 360 + 6 − 90 − 30 +
180 + 180 + 15 − 180 + 180 + 15 −
360 − 180 + 30 − 180 + 6 + 30 −
360 + 135 + 360 − 30 + 180 + 180 −
15 + 180 − 180 − 540 − 180 − 720 −
45 + 30 + 180 + 6 − 180 + 180 −
360 − 180 + 15 − 15 − 45 − 30 +
15 − 360 + 30 + 360 + 180 ℏ^7 + O[ℏ]^8