

Pensieve header: Trees and Wheels for the Borromean Rings.

```

SetDirectory["C:\\drorbn\\AcademicPensieve\\2013-03"];
<< FreeLie.m

Domain[f_List] := First /@ f;
f_ \ key_ := DeleteCases[f, key → _];
f_ \ keys_List := Fold[#1 \ #2 &, f, keys];
f1_List ≡ f2_List := Domain[f1] === Domain[f2] && (And @@ (
  ((# /. f1) ≡ (# /. f2)) & /@ Domain[f1]
));
(* LieDerivation[der_] [f_List] := MapAt[LieDerivation[der], f, {All, 2}]; *)
LieMorphism[mor_] [f_List] := MapAt[LieMorphism[mor], f, {All, 2}];
M /: M[λ1_, ω1_] ∪ M[λ2_, ω2_] := M[λ1 ∪ λ2, ω1 + ω2];
M[λ1_, ω1_] ≡ M[λ2_, ω2_] := (λ1 ≡ λ2) && (ω1 ≡ ω2);

tm[u_, v_, w_] [λ_List] := λ // LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩];
tm[u_, v_, w_] [M[λ_, ω_]] := LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩] /@ M[λ, ω];
hm[x_, y_, z_] [λ_List] := Union[λ \ {x, y}, {z → BCH[x /. λ, y /. λ]}];
hm[x_, y_, z_] [M[λ_, ω_]] := M[λ // hm[x, y, z], ω];
RC[u_, λx_LieSeries, ub_] [ser_] := StableApply[
  LieMorphism[⟨u⟩ → Ad[λx][⟨ub⟩]],
  ser
];
RC[u_, λx_LieSeries] [ser_] :=
  ser // RC[⟨u⟩, λx, ⟨"v"⟩] // LieMorphism[⟨"v"⟩ → ⟨u⟩];
J[u_, λx_] := Module[{s},
  IntegrateCWSeries[
    div[⟨u⟩, λx // RC[⟨u⟩, s λx]] // LieMorphism[⟨u⟩ → Ad[-s λx][⟨u⟩]],
    {s, 0, 1}
  ]
];
tha[u_, x_] [λ_List] := MapAt[RC[⟨u⟩, x /. λ], λ, {All, 2}];
tha[u_, x_] [M[λ_, ω_]] :=
  M[λ // tha[u, x], (ω + J[⟨u⟩, x /. λ]) // RC[⟨u⟩, x /. λ]];
dm[a_, b_, c_] [μ_] := μ // tha[⟨a⟩, b] // tm[⟨a⟩, ⟨b⟩, ⟨c⟩] // hm[a, b, c];
R+[u_, x_] := M[{x → MakeLieSeries[⟨u⟩], u → MakeLieSeries[0]}, MakeCWSeries[0]];
R-[u_, x_] := M[{x → MakeLieSeries[-⟨u⟩], u → MakeLieSeries[0]}, MakeCWSeries[0]];
μ0 = R-[r, 6] ∪ R+[2, 4] ∪ R-[g, 9] ∪ R+[5, 7] ∪ R-[b, 3] ∪ R+[8, 1]
M[{1 → LS[8, 0, 0], 2 → LS[0, 0, 0], 3 → LS[-b, 0, 0], 4 → LS[2, 0, 0],
  5 → LS[0, 0, 0], 6 → LS[-r, 0, 0], 7 → LS[5, 0, 0], 8 → LS[0, 0, 0],
  9 → LS[-g, 0, 0], b → LS[0, 0, 0], g → LS[0, 0, 0], r → LS[0, 0, 0]}, CWS[0, 0, 0]]

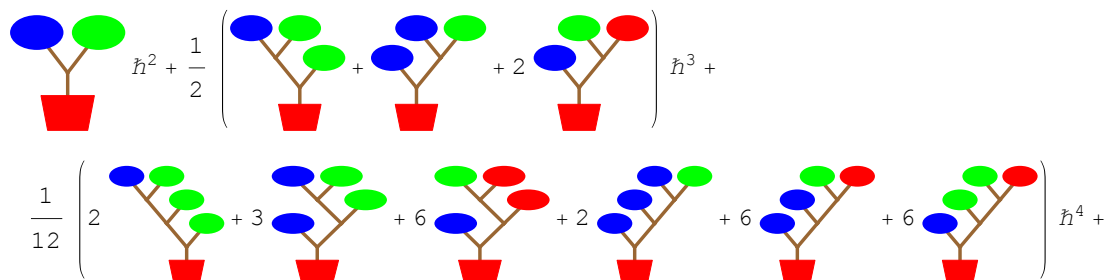
```

```
Do[μ0 = μ0 // dm[r, k, r], {k, 1, 3}];
Do[μ0 = μ0 // dm[g, k, g], {k, 4, 6}];
Do[μ0 = μ0 // dm[b, k, b], {k, 7, 9}];
μ0
```

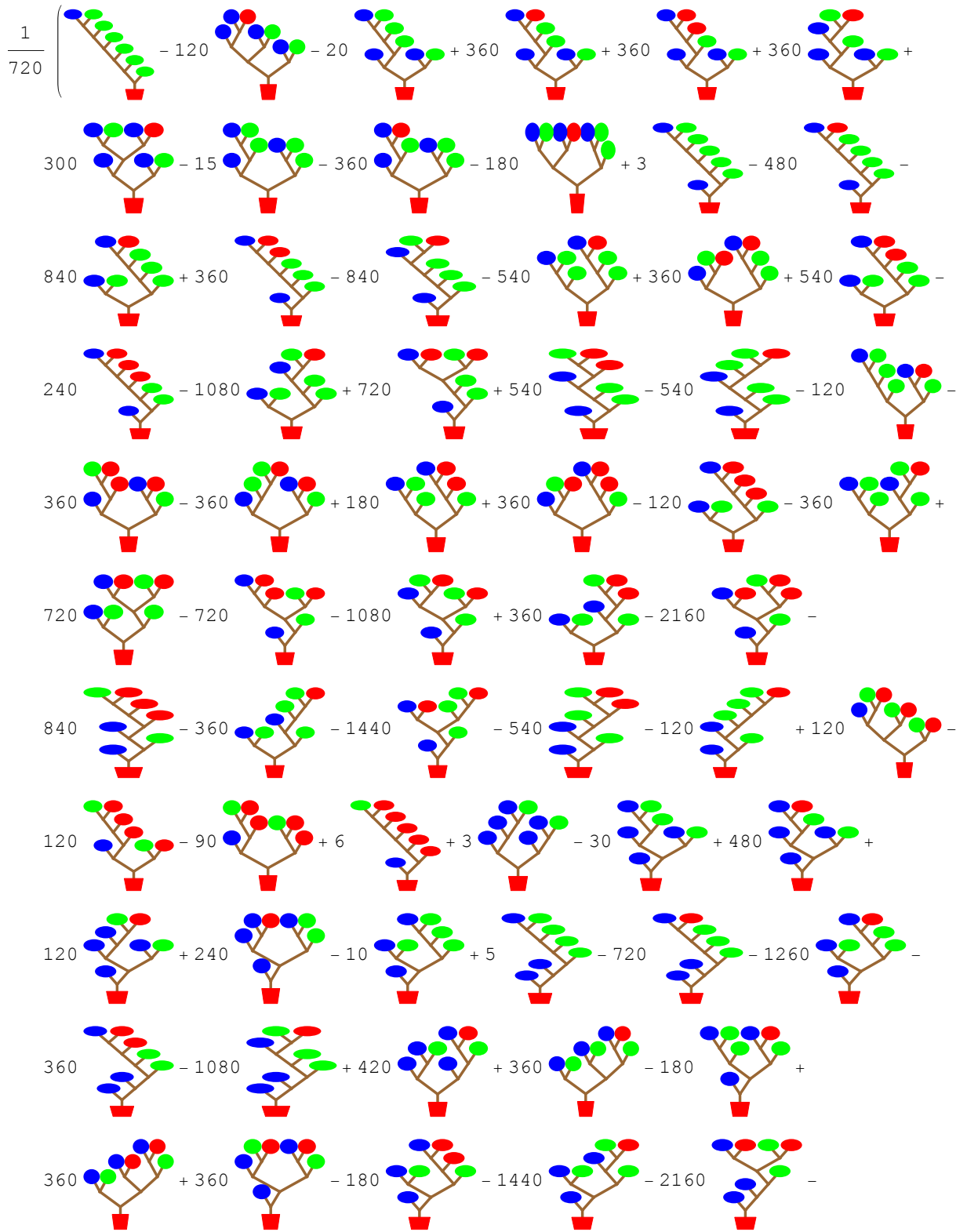
$$M\left[\left\{b \rightarrow LS\left[0, \overline{gr}, \frac{1}{2} \overline{ggr} + \overline{brg} + \frac{1}{2} \overline{grr}\right], g \rightarrow LS\left[0, -\overline{br}, \frac{1}{2} \overline{bbr} - \overline{bgr} - \overline{brg} + \frac{1}{2} \overline{brr}\right], r \rightarrow LS\left[0, \overline{bg}, \frac{1}{2} \overline{bbg} + \overline{bgr} + \frac{1}{2} \overline{bgg}\right]\right\}, CWS[0, 0, 2 CW[bgr]]\right]$$

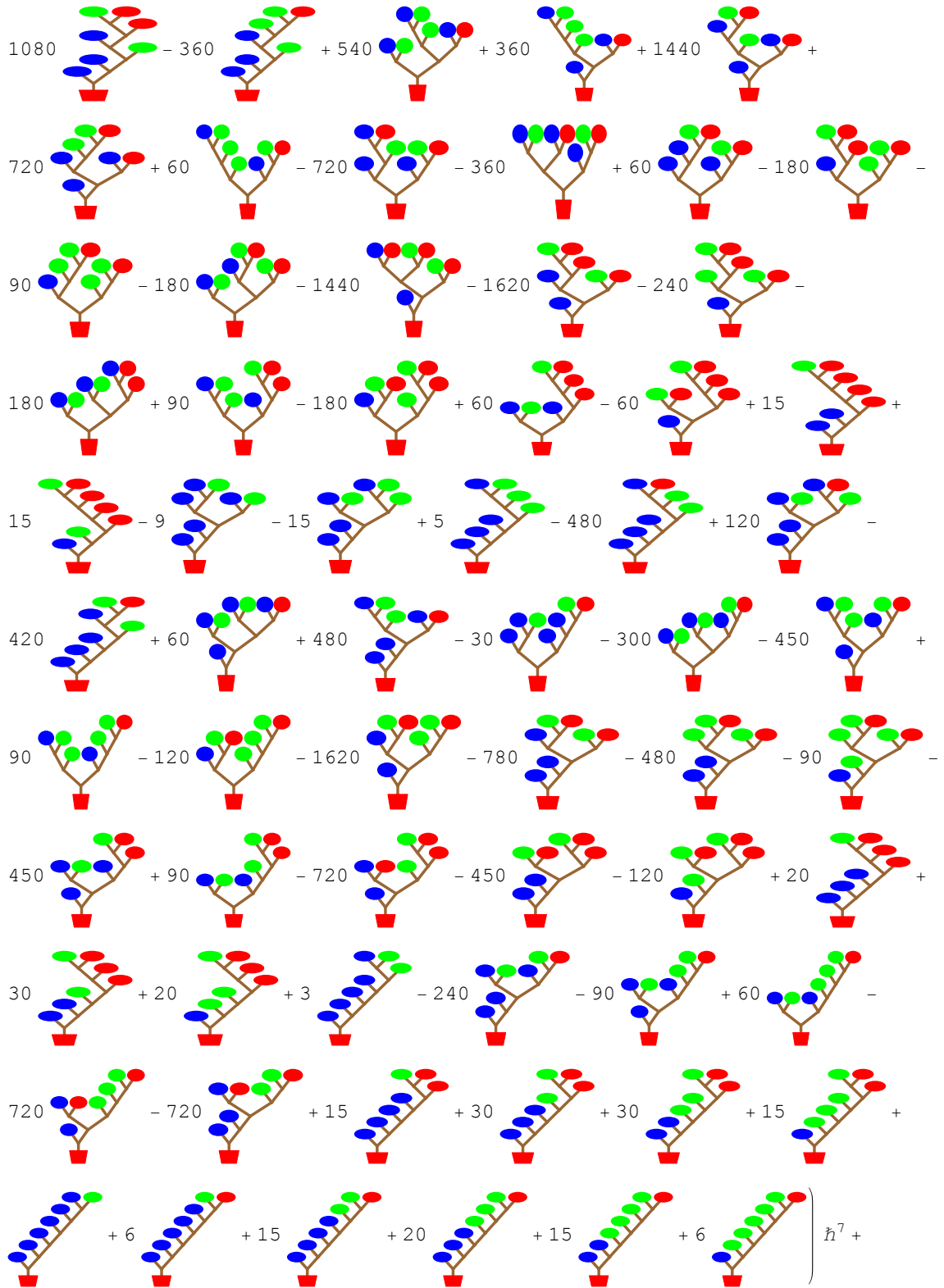
## Trees

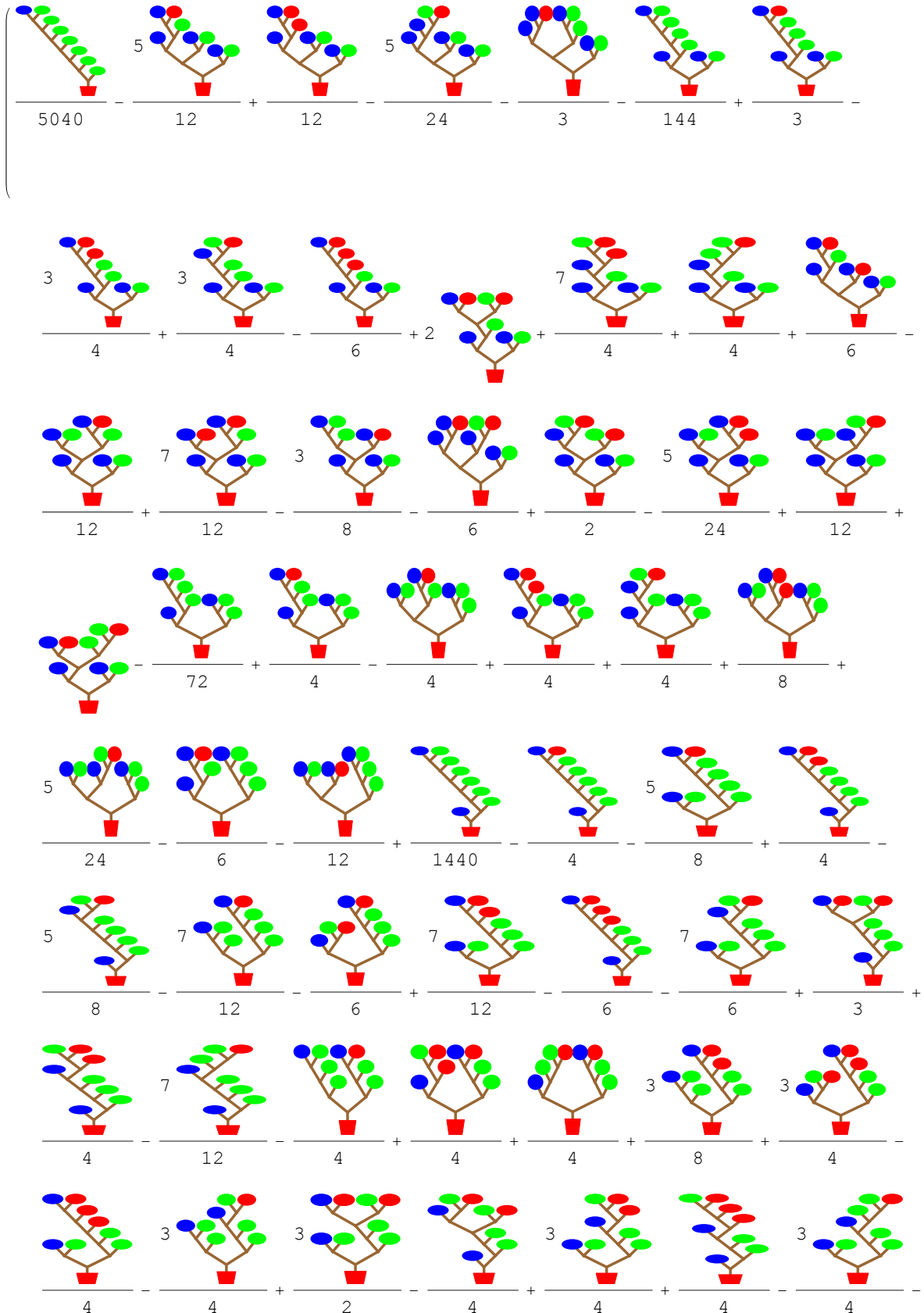
```
trees = Table[(r /. First[μ0])@k, {k, 8}];
t1 = Series[(List@@trees // . w_LW => B@@Reverse[LyndonFactorization[w]] /.
  B[s_] => s /. t_B => Tree[t]).ħ^Range[Length[trees]],
  {ħ, 0, Length[trees]}
] /. {"r" -> r, "g" -> g, "b" -> b};
t1 /. t_Tree => TreeForm[t,
  VertexRenderingFunction -> (Switch[#2,
    Tree, {
      Red,
      Polygon[
        {{-0.4, 0.4} - #1, {0.4, 0.4} - #1, {0.3, -0.4} - #1, {-0.3, -0.4} - #1}
      ],
      B, {},
      _, {
        ReleaseHold[#2 /. {r -> Red, g -> Green, b -> Blue}],
        Disk[-#1, 0.4]
      }
    ] &),
  EdgeRenderingFunction -> ({
    Brown, Thickness[0.03],
    Line[-#]
  } &),
  PlotRangePadding -> 0, ImageSize -> 60, AspectRatio -> 1
]
```

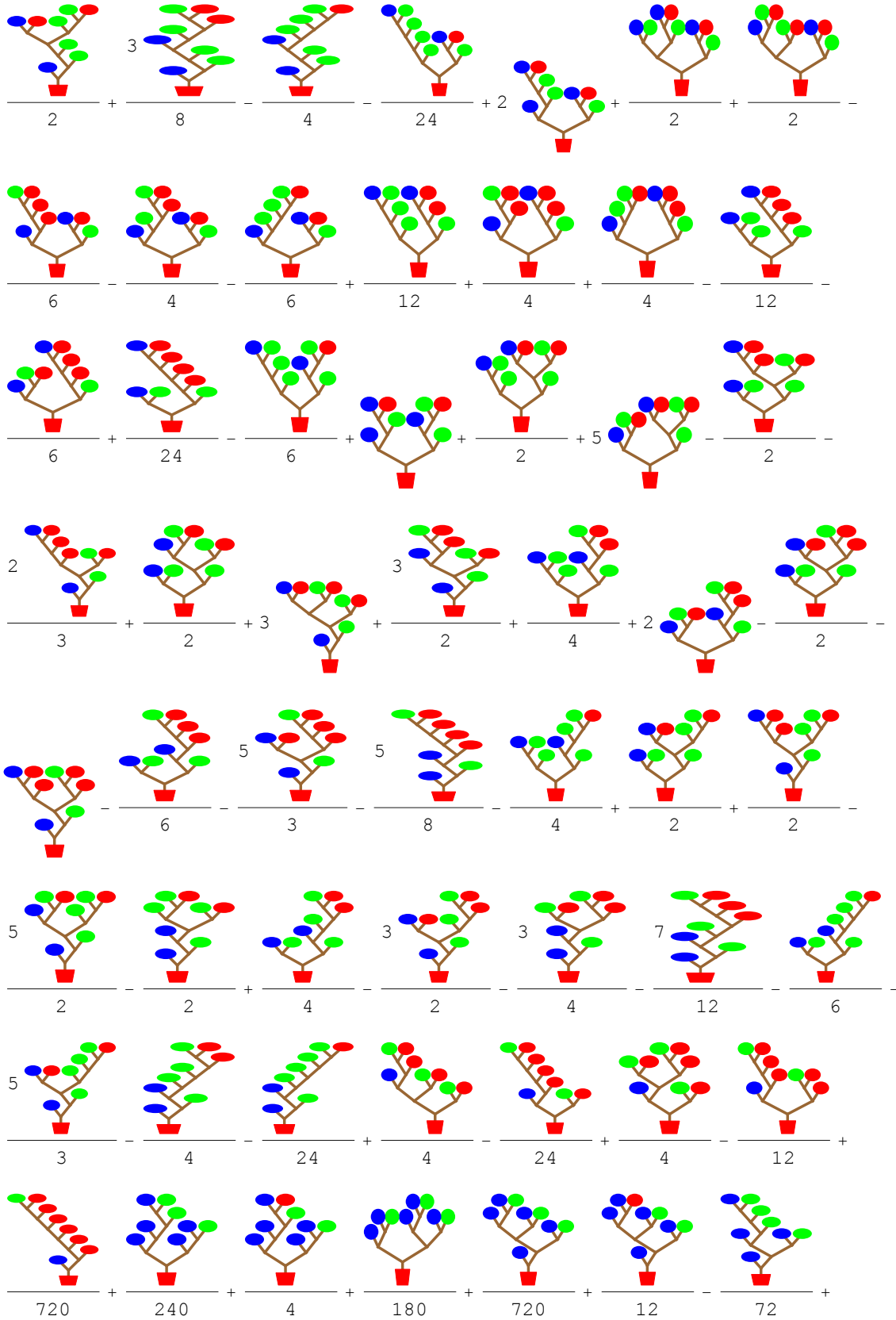


$$\begin{aligned}
 & \frac{1}{24} \left( \begin{array}{c}
 \text{Tree 1} - 2 \text{Tree 2} + 2 \text{Tree 3} - 48 \text{Tree 4} - 24 \text{Tree 5} - \\
 24 \text{Tree 6} - 12 \text{Tree 7} + 4 \text{Tree 8} + 2 \text{Tree 9} + 12 \text{Tree 10} + \\
 6 \text{Tree 11} + 6 \text{Tree 12} + \text{Tree 13} + 4 \text{Tree 14} + 6 \text{Tree 15} + 4 \text{Tree 16}
 \end{array} \right) \hbar^5 + \\
 & \frac{1}{720} \left( \begin{array}{c}
 6 \text{Tree 17} - 60 \text{Tree 18} - 720 \text{Tree 19} + 15 \text{Tree 20} - 720 \text{Tree 21} - \\
 1080 \text{Tree 22} - 1080 \text{Tree 23} - 360 \text{Tree 24} - 720 \text{Tree 25} + 360 \text{Tree 26} - \\
 720 \text{Tree 27} - 2880 \text{Tree 28} - 1080 \text{Tree 29} - 360 \text{Tree 30} - 360 \text{Tree 31} + \\
 30 \text{Tree 32} - 30 \text{Tree 33} - 30 \text{Tree 34} + 20 \text{Tree 35} - 720 \text{Tree 36} + 360 \text{Tree 37} - \\
 720 \text{Tree 38} + 360 \text{Tree 39} + 720 \text{Tree 40} + 180 \text{Tree 41} - 360 \text{Tree 42} - \\
 900 \text{Tree 43} - 240 \text{Tree 44} + 180 \text{Tree 45} - 180 \text{Tree 46} + 60 \text{Tree 47} + 60 \text{Tree 48} + \\
 15 \text{Tree 49} - 180 \text{Tree 50} + 180 \text{Tree 51} - 1440 \text{Tree 52} + 60 \text{Tree 53} + 90 \text{Tree 54} + \\
 60 \text{Tree 55} + 6 \text{Tree 56} + 30 \text{Tree 57} + 60 \text{Tree 58} + 60 \text{Tree 59} + 30 \text{Tree 60}
 \end{array} \right) \hbar^6 +
 \end{aligned}$$









$$\begin{array}{c}
 \begin{array}{c} \text{Tree 1} \\ \text{Tree 2} \end{array} + \begin{array}{c} \text{Tree 3} \\ \text{Tree 4} \end{array} + \frac{5 \begin{array}{c} \text{Tree 5} \\ \text{Tree 6} \end{array}}{24} - \frac{5 \begin{array}{c} \text{Tree 7} \\ \text{Tree 8} \end{array}}{24} + \frac{2 \begin{array}{c} \text{Tree 9} \\ \text{Tree 10} \end{array}}{3} + \frac{\begin{array}{c} \text{Tree 11} \\ \text{Tree 12} \end{array}}{4} + \frac{\begin{array}{c} \text{Tree 13} \\ \text{Tree 14} \end{array}}{12} -
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \text{Tree 15} \\ \text{Tree 16} \end{array} - \begin{array}{c} \text{Tree 17} \\ \text{Tree 18} \end{array} + \begin{array}{c} \text{Tree 19} \\ \text{Tree 20} \end{array} - \begin{array}{c} \text{Tree 21} \\ \text{Tree 22} \end{array} + \begin{array}{c} \text{Tree 23} \\ \text{Tree 24} \end{array} - \begin{array}{c} \text{Tree 25} \\ \text{Tree 26} \end{array} + \frac{\begin{array}{c} \text{Tree 27} \\ \text{Tree 28} \end{array}}{144} - \frac{\begin{array}{c} \text{Tree 29} \\ \text{Tree 30} \end{array}}{6} -
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \text{Tree 31} \\ \text{Tree 32} \end{array} + \begin{array}{c} \text{Tree 33} \\ \text{Tree 34} \end{array} - \frac{7 \begin{array}{c} \text{Tree 35} \\ \text{Tree 36} \end{array}}{12} - \frac{17 \begin{array}{c} \text{Tree 37} \\ \text{Tree 38} \end{array}}{12} + \frac{9 \begin{array}{c} \text{Tree 39} \\ \text{Tree 40} \end{array}}{2} + \frac{7 \begin{array}{c} \text{Tree 41} \\ \text{Tree 42} \end{array}}{2} -
 \end{array}$$

$$\begin{array}{c}
 \frac{7 \begin{array}{c} \text{Tree 43} \\ \text{Tree 44} \end{array}}{6} + \frac{3 \begin{array}{c} \text{Tree 45} \\ \text{Tree 46} \end{array}}{8} - \frac{3 \begin{array}{c} \text{Tree 47} \\ \text{Tree 48} \end{array}}{4} - \frac{9 \begin{array}{c} \text{Tree 49} \\ \text{Tree 50} \end{array}}{8} + \frac{13 \begin{array}{c} \text{Tree 51} \\ \text{Tree 52} \end{array}}{2} + \frac{11 \begin{array}{c} \text{Tree 53} \\ \text{Tree 54} \end{array}}{4} +
 \end{array}$$

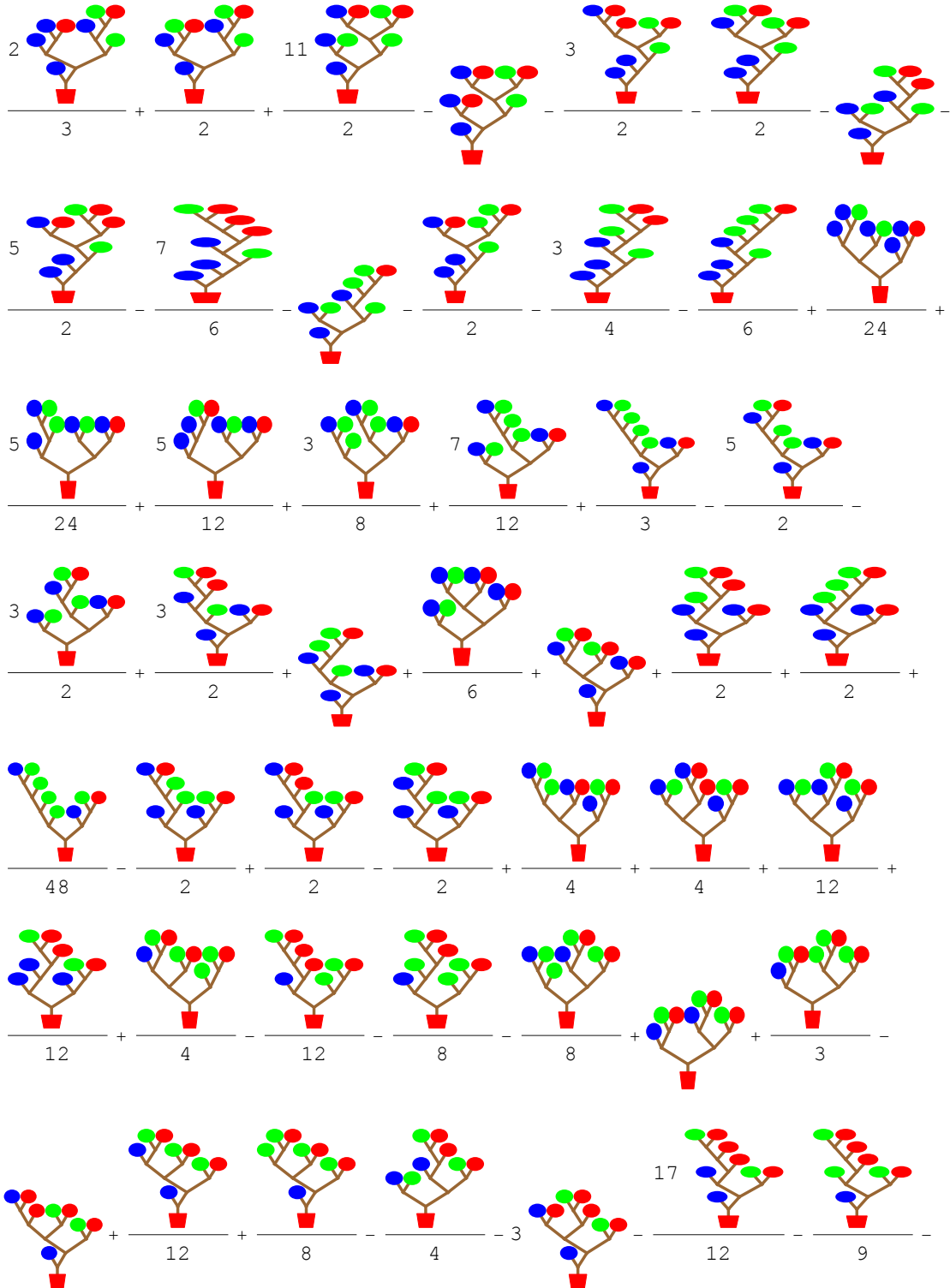
$$\begin{array}{c}
 \frac{\begin{array}{c} \text{Tree 55} \\ \text{Tree 56} \end{array}}{4} + \frac{43 \begin{array}{c} \text{Tree 57} \\ \text{Tree 58} \end{array}}{8} - \frac{\begin{array}{c} \text{Tree 59} \\ \text{Tree 60} \end{array}}{2} - \frac{\begin{array}{c} \text{Tree 61} \\ \text{Tree 62} \end{array}}{6} - \frac{5 \begin{array}{c} \text{Tree 63} \\ \text{Tree 64} \end{array}}{2} + \frac{11 \begin{array}{c} \text{Tree 65} \\ \text{Tree 66} \end{array}}{2} +
 \end{array}$$

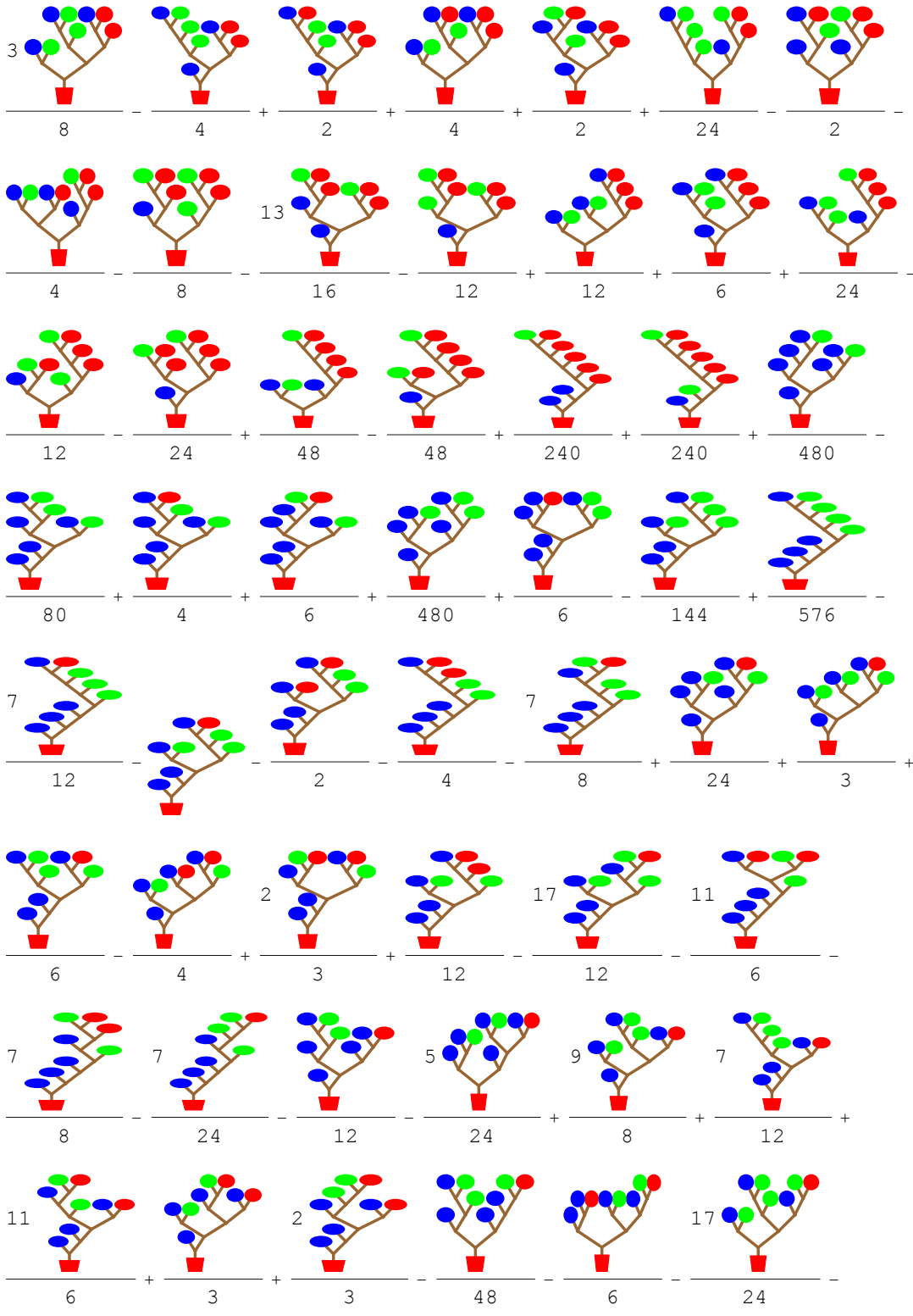
$$\begin{array}{c}
 \frac{3 \begin{array}{c} \text{Tree 67} \\ \text{Tree 68} \end{array}}{4} - \frac{3 \begin{array}{c} \text{Tree 69} \\ \text{Tree 70} \end{array}}{4} + \frac{7 \begin{array}{c} \text{Tree 71} \\ \text{Tree 72} \end{array}}{24} + \frac{7 \begin{array}{c} \text{Tree 73} \\ \text{Tree 74} \end{array}}{12} + \frac{\begin{array}{c} \text{Tree 75} \\ \text{Tree 76} \end{array}}{4} + \frac{\begin{array}{c} \text{Tree 77} \\ \text{Tree 78} \end{array}}{12} - 4 \begin{array}{c} \text{Tree 79} \\ \text{Tree 80} \end{array} +
 \end{array}$$

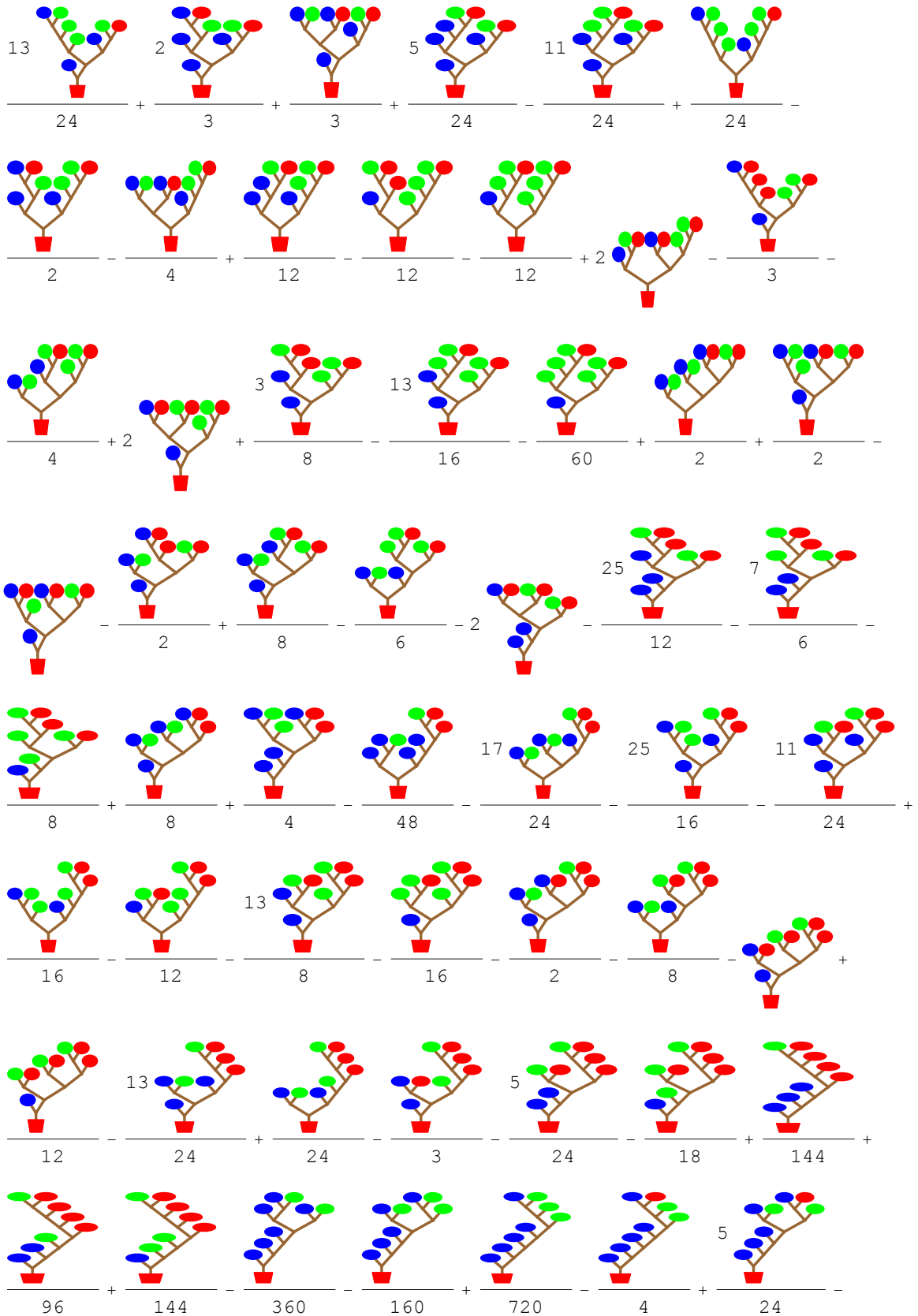
$$\begin{array}{c}
 \begin{array}{c} \text{Tree 81} \\ \text{Tree 82} \end{array} + \begin{array}{c} \text{Tree 83} \\ \text{Tree 84} \end{array} - \begin{array}{c} \text{Tree 85} \\ \text{Tree 86} \end{array} + \frac{3 \begin{array}{c} \text{Tree 87} \\ \text{Tree 88} \end{array}}{4} - \frac{3 \begin{array}{c} \text{Tree 89} \\ \text{Tree 90} \end{array}}{4} - \frac{7 \begin{array}{c} \text{Tree 91} \\ \text{Tree 92} \end{array}}{24} + \frac{3 \begin{array}{c} \text{Tree 93} \\ \text{Tree 94} \end{array}}{4} +
 \end{array}$$

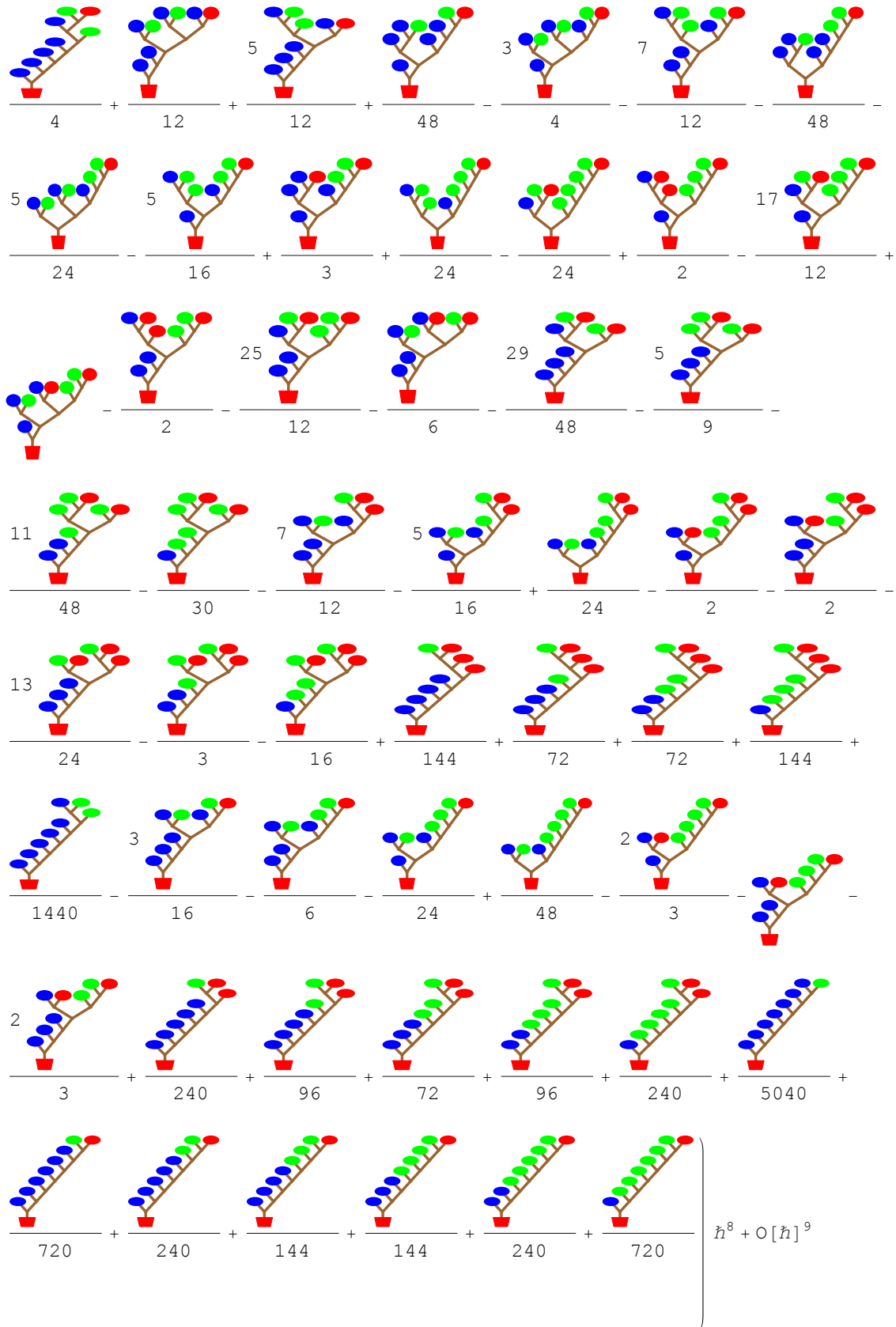
$$\begin{array}{c}
 \frac{13 \begin{array}{c} \text{Tree 95} \\ \text{Tree 96} \end{array}}{8} - \frac{\begin{array}{c} \text{Tree 97} \\ \text{Tree 98} \end{array}}{4} + \frac{\begin{array}{c} \text{Tree 99} \\ \text{Tree 100} \end{array}}{4} + \frac{\begin{array}{c} \text{Tree 101} \\ \text{Tree 102} \end{array}}{12} + \frac{\begin{array}{c} \text{Tree 103} \\ \text{Tree 104} \end{array}}{3} - \frac{3 \begin{array}{c} \text{Tree 105} \\ \text{Tree 106} \end{array}}{2} - 2 \begin{array}{c} \text{Tree 107} \\ \text{Tree 108} \end{array} +
 \end{array}$$











# Wheels

```

n = 7;
wheels = Table[Last[μ0]@k, {k, n}];
SetOptions[Rasterize, {RasterSize → 256, ImageSize → 256}];
Collect[
  Expand[(Plus @@ wheels)] /.
    CW[s_String] := ħStringLength[s] Show[ImageCrop[PieChart3D[
      Table[1, {StringLength[s]}],
      ChartStyle := (Characters[s] /. {"r" → Red, "g" → Green, "b" → Blue}),
      SectorOrigin → {{RandomReal[{0, 2 π}], "Counterclockwise"}, 1},
      ChartBaseStyle → EdgeForm[{Thickness[0.03], Black}],
      ChartElementFunction → "ProfileSector3D",
      ImagePadding → 0, ImageMargins → 0, PlotRangePadding → 0
    ]], ImageSize → 52],
  ħ, Factor] + O[ħ]n+1

```

$$\begin{aligned}
 & 2 \text{ (wheel)} \hbar^3 + \left( \text{ (wheel)} + \text{ (wheel)} - \text{ (wheel)} - \text{ (wheel)} + \text{ (wheel)} - \text{ (wheel)} \right) \hbar^4 + \\
 & \frac{1}{6} \left( 3 \text{ (wheel)} + 2 \text{ (wheel)} - 3 \text{ (wheel)} + 3 \text{ (wheel)} + 3 \text{ (wheel)} + 3 \text{ (wheel)} + \right. \\
 & 3 \text{ (wheel)} + 3 \text{ (wheel)} + 3 \text{ (wheel)} + 2 \text{ (wheel)} + 3 \text{ (wheel)} - 9 \text{ (wheel)} - \\
 & \left. 3 \text{ (wheel)} + 3 \text{ (wheel)} - 9 \text{ (wheel)} - 3 \text{ (wheel)} + 2 \text{ (wheel)} - 9 \text{ (wheel)} \right) \hbar^5 + \\
 & \frac{1}{12} \left( 12 \text{ (wheel)} - 9 \text{ (wheel)} + \text{ (wheel)} - 2 \text{ (wheel)} - 2 \text{ (wheel)} - 2 \text{ (wheel)} - 12 \text{ (wheel)} - \right. \\
 & 9 \text{ (wheel)} + 2 \text{ (wheel)} - 3 \text{ (wheel)} + 3 \text{ (wheel)} + 9 \text{ (wheel)} + 12 \text{ (wheel)} + 12 \text{ (wheel)} - \\
 & 2 \text{ (wheel)} + 12 \text{ (wheel)} - 15 \text{ (wheel)} + 12 \text{ (wheel)} + \text{ (wheel)} - 3 \text{ (wheel)} + 2 \text{ (wheel)} - \\
 & 15 \text{ (wheel)} + 3 \text{ (wheel)} + 12 \text{ (wheel)} - 9 \text{ (wheel)} - 12 \text{ (wheel)} - 12 \text{ (wheel)} + \\
 & 12 \text{ (wheel)} - 2 \text{ (wheel)} - 2 \text{ (wheel)} + 3 \text{ (wheel)} - 2 \text{ (wheel)} + 12 \text{ (wheel)} + \\
 & 9 \text{ (wheel)} + 2 \text{ (wheel)} + 2 \text{ (wheel)} - 2 \text{ (wheel)} + 3 \text{ (wheel)} - 2 \text{ (wheel)} + 2 \text{ (wheel)} - \\
 & 2 \text{ (wheel)} - 12 \text{ (wheel)} + 2 \text{ (wheel)} + 3 \text{ (wheel)} - 2 \text{ (wheel)} + 9 \text{ (wheel)} - 3 \text{ (wheel)} + \\
 & \left. 12 \text{ (wheel)} + 3 \text{ (wheel)} - 15 \text{ (wheel)} - 12 \text{ (wheel)} + 12 \text{ (wheel)} - 48 \text{ (wheel)} + \right)
 \end{aligned}$$



$$\begin{aligned}
 & 180 \text{ (diagram)} + 360 \text{ (diagram)} - 180 \text{ (diagram)} - 90 \text{ (diagram)} + 180 \text{ (diagram)} + 20 \text{ (diagram)} - \\
 & 1080 \text{ (diagram)} + 225 \text{ (diagram)} + 30 \text{ (diagram)} + 180 \text{ (diagram)} + 1080 \text{ (diagram)} + 180 \text{ (diagram)} + \\
 & 30 \text{ (diagram)} + 30 \text{ (diagram)} + 180 \text{ (diagram)} - 225 \text{ (diagram)} - 90 \text{ (diagram)} - 180 \text{ (diagram)} + \\
 & 30 \text{ (diagram)} - 180 \text{ (diagram)} + 15 \text{ (diagram)} + 180 \text{ (diagram)} + 360 \text{ (diagram)} + 180 \text{ (diagram)} + \\
 & 30 \text{ (diagram)} + 180 \text{ (diagram)} + 30 \text{ (diagram)} + 15 \text{ (diagram)} - 180 \text{ (diagram)} + 180 \text{ (diagram)} - \\
 & 45 \text{ (diagram)} + 30 \text{ (diagram)} - 45 \text{ (diagram)} + 360 \text{ (diagram)} + 6 \text{ (diagram)} - 90 \text{ (diagram)} - 30 \text{ (diagram)} + \\
 & 180 \text{ (diagram)} + 180 \text{ (diagram)} + 15 \text{ (diagram)} - 180 \text{ (diagram)} + 180 \text{ (diagram)} + 15 \text{ (diagram)} - \\
 & 360 \text{ (diagram)} - 180 \text{ (diagram)} + 30 \text{ (diagram)} - 180 \text{ (diagram)} + 6 \text{ (diagram)} + 30 \text{ (diagram)} - \\
 & 360 \text{ (diagram)} + 135 \text{ (diagram)} + 360 \text{ (diagram)} - 30 \text{ (diagram)} + 180 \text{ (diagram)} + 180 \text{ (diagram)} - \\
 & 15 \text{ (diagram)} + 180 \text{ (diagram)} - 180 \text{ (diagram)} - 540 \text{ (diagram)} - 180 \text{ (diagram)} - 720 \text{ (diagram)} - \\
 & 45 \text{ (diagram)} + 30 \text{ (diagram)} + 180 \text{ (diagram)} + 6 \text{ (diagram)} - 180 \text{ (diagram)} + 180 \text{ (diagram)} - \\
 & 360 \text{ (diagram)} - 180 \text{ (diagram)} + 15 \text{ (diagram)} - 15 \text{ (diagram)} - 45 \text{ (diagram)} - 30 \text{ (diagram)} + \\
 & 15 \text{ (diagram)} - 360 \text{ (diagram)} + 30 \text{ (diagram)} + 360 \text{ (diagram)} + 180 \text{ (diagram)} \Big) \hbar^7 + O[\hbar]^8
 \end{aligned}$$