

```
In[1]:= grow[f_fl] := Append[f, 1] +  $\sum_{i=1}^{\text{Length}[f]}$  MapAt[(# + 1) &, f, i];
      grow[expr_] := Expand[expr /. f_fl => grow[f]]
```

```
In[3]:= fl[] // grow
```

```
Out[3]= fl[1]
```

```
In[4]:= fl[] // grow // grow
```

```
Out[4]= fl[2] + fl[1, 1]
```

```
In[5]:= fl[] // grow // grow // grow
```

```
Out[5]= fl[3] + fl[1, 2] + 2 fl[2, 1] + fl[1, 1, 1]
```

```
In[6]:= sf[n_] := Module[{s = fl[]}, s +  $\sum_{k=1}^n$  Expand[ $\frac{s = \text{grow}[s]}{k!}$ ]];
```

```
In[7]:= sf[3]
```

```
Out[7]= fl[] + fl[1] +  $\frac{fl[2]}{2}$  +  $\frac{fl[3]}{6}$  +  $\frac{1}{2}$  fl[1, 1] +  $\frac{1}{6}$  fl[1, 2] +  $\frac{1}{3}$  fl[2, 1] +  $\frac{1}{6}$  fl[1, 1, 1]
```

```
In[8]:= sf[2]
```

```
Out[8]= fl[] + fl[1] +  $\frac{fl[2]}{2}$  +  $\frac{1}{2}$  fl[1, 1]
```

```
In[9]:= cop[f_fl] := Plus @@ (
```

```
      (Pick[fl1@@f, #, 1] Pick[fl2@@f, #, 2]) & /@ Tuples[{1, 2}, Length[f]]
    );
```

```
      cop[expr_] := Expand[expr /. f_fl => cop[f]]
```

```
In[11]:= cop[fl[1, 2]]
```

```
Out[11]= fl1[1, 2] fl2[] + fl1[2] fl2[1] + fl1[1] fl2[2] + fl1[] fl2[1, 2]
```

```
In[12]:= cop[sf[2]]
```

```
Out[12]= fl1[] fl2[] + fl1[1] fl2[] +  $\frac{1}{2}$  fl1[2] fl2[] +  $\frac{1}{2}$  fl1[1, 1] fl2[] +
      fl1[] fl2[1] + fl1[1] fl2[1] +  $\frac{1}{2}$  fl1[] fl2[2] +  $\frac{1}{2}$  fl1[] fl2[1, 1]
```

```
In[13]:= sfsf[n_] := Module[{s},
  s = sf[n];
  Sum[Expand[Sum[Times[
    s /. f_f1 -> If[(Plus@@f) == d1, f11@@f, 0],
    s /. f_f1 -> If[(Plus@@f) == k - d1, f12@@f, 0]
  ]], {d1, 0, k}], {k, 0, n}
]
```

```
In[14]:= cop[sf[2]] == sfsf[2]
```

```
Out[14]= True
```

```
In[15]:= cop[sf[3]]
```

```
Out[15]= f11[] f12[] + f11[1] f12[] +  $\frac{1}{2}$  f11[2] f12[] +  $\frac{1}{6}$  f11[3] f12[] +
 $\frac{1}{2}$  f11[1, 1] f12[] +  $\frac{1}{6}$  f11[1, 2] f12[] +  $\frac{1}{3}$  f11[2, 1] f12[] +  $\frac{1}{6}$  f11[1, 1, 1] f12[] +
f11[] f12[1] + f11[1] f12[1] +  $\frac{1}{2}$  f11[2] f12[1] +  $\frac{1}{2}$  f11[1, 1] f12[1] +
 $\frac{1}{2}$  f11[] f12[2] +  $\frac{1}{2}$  f11[1] f12[2] +  $\frac{1}{6}$  f11[] f12[3] +  $\frac{1}{2}$  f11[] f12[1, 1] +
 $\frac{1}{2}$  f11[1] f12[1, 1] +  $\frac{1}{6}$  f11[] f12[1, 2] +  $\frac{1}{3}$  f11[] f12[2, 1] +  $\frac{1}{6}$  f11[] f12[1, 1, 1]
```

```
In[16]:= sfsf[3]
```

```
Out[16]= f11[] f12[] + f11[1] f12[] +  $\frac{1}{2}$  f11[2] f12[] +  $\frac{1}{6}$  f11[3] f12[] +
 $\frac{1}{2}$  f11[1, 1] f12[] +  $\frac{1}{6}$  f11[1, 2] f12[] +  $\frac{1}{3}$  f11[2, 1] f12[] +  $\frac{1}{6}$  f11[1, 1, 1] f12[] +
f11[] f12[1] + f11[1] f12[1] +  $\frac{1}{2}$  f11[2] f12[1] +  $\frac{1}{2}$  f11[1, 1] f12[1] +
 $\frac{1}{2}$  f11[] f12[2] +  $\frac{1}{2}$  f11[1] f12[2] +  $\frac{1}{6}$  f11[] f12[3] +  $\frac{1}{2}$  f11[] f12[1, 1] +
 $\frac{1}{2}$  f11[1] f12[1, 1] +  $\frac{1}{6}$  f11[] f12[1, 2] +  $\frac{1}{3}$  f11[] f12[2, 1] +  $\frac{1}{6}$  f11[] f12[1, 1, 1]
```

```
In[17]:= cop[sf[3]] == sfsf[3]
```

```
Out[17]= True
```

```
In[18]:= cop[sf[7]] == sfsf[7]
```

```
Out[18]= True
```

In[19]= **sf[4]**

$$\begin{aligned} \text{Out[19]= } & f1[] + f1[1] + \frac{f1[2]}{2} + \frac{f1[3]}{6} + \frac{f1[4]}{24} + \frac{1}{2} f1[1, 1] + \frac{1}{6} f1[1, 2] + \\ & \frac{1}{24} f1[1, 3] + \frac{1}{3} f1[2, 1] + \frac{1}{8} f1[2, 2] + \frac{1}{8} f1[3, 1] + \frac{1}{6} f1[1, 1, 1] + \\ & \frac{1}{24} f1[1, 1, 2] + \frac{1}{12} f1[1, 2, 1] + \frac{1}{8} f1[2, 1, 1] + \frac{1}{24} f1[1, 1, 1, 1] \end{aligned}$$

Unprotect[NonCommutativeMultiply];

(a_Plus) ** b_ := (# ** b) & /@ a;

a_ ** (b_Plus) := (a ** #) & /@ b;

(a_.*f1[b___]) ** (c_.*f1[d___]) := a c f1[b, d];