Cheat Sheet J - Verification

Pensieve header: Cheat Sheet J Verification; continued pensieve://2013-04/.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2013-03"];
<< FreeLie.m;
tm[u_, v_, w_] := LieMorphism[⟨u⟩ → ⟨w⟩, ⟨v⟩ → ⟨w⟩];
CC[u_, γ_LieSeries] := LieMorphism[u → Ad[γ][u]];
RC[u_, γ_LieSeries, ub_][ser_] :=
 StableApply[LieMorphism[⟨u⟩ → Ad[γ][⟨ub⟩]], ser];
RC[u_, γ_LieSeries][ser_] := ser // RC[u, γ, ⟨υ⟩] // LieMorphism[⟨υ⟩ → ⟨u⟩];
Print /@ {{t = ⟨"t"⟩, u = ⟨"u"⟩, v = ⟨"v"⟩, w = ⟨"w"⟩},
   α = RandomLieSeries[{t, u, v}],
   β = RandomLieSeries[{t, u, v}],
   γ = RandomLieSeries[{t, u, v}]
  };
$SeriesShowDegree = 3; $SeriesCompareDegree = 6;
```

```
{⟨t⟩, ⟨u⟩, ⟨v⟩, ⟨w⟩}
```

$$\text{LS}\left[2\,t-v,\ -\frac{\overline{tu}}{2}+\overline{tv}-2\,\overline{uv},\ -\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{4}{3}\,\overline{t\,\overline{tv}}-\frac{3}{2}\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}-\frac{1}{6}\,\overline{\overline{tu}\,u}-\overline{\overline{tv}\,u}+\overline{\overline{tv}\,v}+2\,\overline{\overline{uv}\,v}\right]$$

$$\text{LS}\left[t-u-v,\ \overline{tu}-\frac{\overline{tv}}{2}+\overline{uv},\ -\frac{3}{2}\,\overline{t\,\overline{tu}}+\frac{5}{6}\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}-\frac{1}{6}\,\overline{\overline{tu}\,u}-\overline{\overline{tv}\,u}-\frac{3}{2}\,\overline{\overline{tv}\,v}-\overline{\overline{uv}\,v}\right]$$

$$\text{LS}\left[t-2\,u+v,\ -\frac{3\,\overline{tu}}{2}+\overline{tv}+\frac{\overline{uv}}{2},\right.$$
$$\left.-\frac{3}{2}\,\overline{t\,\overline{tu}}+\frac{4}{3}\,\overline{t\,\overline{tv}}-\frac{1}{6}\,\overline{t\,\overline{uv}}+\frac{1}{6}\,\overline{u\,\overline{uv}}+\frac{2}{3}\,\overline{\overline{tu}\,u}-\frac{7}{6}\,\overline{\overline{tv}\,u}+\overline{\overline{tv}\,v}+\frac{3}{2}\,\overline{\overline{uv}\,v}\right]$$

```
ad[u_, γ_LieSeries] := LieDerivation[u → b[γ, u]];
ϵ /: ϵ^2 = 0;
```

- Some preliminary testing

```
(α // RC[u, α] // CC[u, -α]) ≡ α
```

```
True
```

1. The Definition of J

$$\texttt{J[u\_, γ\_] := } \int_0^1 \texttt{(div[u, γ // RC[u, s γ]] // CC[u, -s γ]) ds}$$

```
J[u, α][{4}]
```

$$\text{CWS}\Big[-2\,\text{CW}[u],\ 2\,\text{CW}[tu]-3\,\text{CW}[uv],\ -\frac{8\,\text{CW}[ttu]}{3}+3\,\text{CW}[tuu]+\frac{3\,\text{CW}[tuv]}{2}+\frac{2\,\text{CW}[tvu]}{3}-\frac{5\,\text{CW}[uuv]}{2}-\frac{2\,\text{CW}[uvv]}{3},\ \frac{59\,\text{CW}[tttu]}{24}+\frac{11\,\text{CW}[ttuu]}{4}+\frac{3\,\text{CW}[ttuv]}{2}+\frac{5\,\text{CW}[ttvu]}{2}-\frac{19\,\text{CW}[tutu]}{2}-\frac{22\,\text{CW}[tutv]}{3}+\frac{31\,\text{CW}[tuuu]}{24}+\frac{77\,\text{CW}[tuuv]}{24}+\frac{23\,\text{CW}[tuvu]}{3}-\frac{3\,\text{CW}[tuvv]}{2}-\frac{25\,\text{CW}[tvuu]}{24}+\frac{13\,\text{CW}[tvuv]}{3}-\frac{23\,\text{CW}[tvvu]}{8}-\frac{\text{CW}[uuuv]}{2}-\frac{97\,\text{CW}[uuvv]}{24}-\frac{11\,\text{CW}[uvuv]}{12}-\frac{13\,\text{CW}[uvvv]}{24}\Big]$$

2. The t equation

```
Print /@ {
    0 → {γ, γw = γ // tm[u, v, w]},
    1 → (t1 = J[w, γw] // RC[w, γw]),
    2 → (t2 = J[u, γ] // tm[u, v, w] // RC[w, γw]),
    3 → (t3 = J[v, γ // RC[u, γ]] // RC[v, γ // RC[u, γ]] // tm[u, v, w]),
    4 → t1 ≡ t2 + t3
   };
```

$$0 \to \Big\{\text{LS}\Big[t+u-v,\ 2\,\overline{tu}+\overline{tv}-\frac{3\,\overline{uv}}{2},\ -\frac{11}{6}\,\overline{t\,\overline{tu}}-\frac{1}{3}\,\overline{t\,\overline{tv}}-\frac{5}{6}\,\overline{t\,\overline{uv}}+2\,\overline{u\,\overline{uv}}+\frac{7}{6}\,\overline{\overline{tu}\,u}-\frac{5}{6}\,\overline{\overline{tv}\,u}-\frac{11}{6}\,\overline{\overline{tv}\,v}-\frac{11}{6}\,\overline{\overline{uv}\,v}\Big],\ \text{LS}\Big[t,\ 3\,\overline{tw},\ -\frac{13}{6}\,\overline{t\,\overline{tw}}-\frac{3}{2}\,\overline{\overline{tw}\,w}\Big]\Big\}$$

$$1 \to \text{CWS}\Big[0,\ 3\,\text{CW}[tw],\ -\frac{2\,\text{CW}[ttw]}{3}+\frac{3\,\text{CW}[tww]}{2}\Big]$$

$$2 \to \text{CWS}\Big[\text{CW}[w],\ \frac{5\,\text{CW}[tw]}{2}+\text{CW}[ww],\ -\frac{2\,\text{CW}[ttw]}{3}-\frac{25\,\text{CW}[tww]}{12}-5\,\text{CW}[www]\Big]$$

$$3 \to \text{CWS}\Big[-\text{CW}[w],\ \frac{\text{CW}[tw]}{2}-\text{CW}[ww],\ \frac{43\,\text{CW}[tww]}{12}+5\,\text{CW}[www]\Big]$$

```
4 → True
```

3. The h equation

```
Print /@ {
    1 → (t1 = J[u, BCH[α, β]]),
    2 → (t2 = J[u, α]),
    3 → (t3 = J[u, β // RC[u, α]] // CC[u, -α]),
    4 → t1 ≡ t2 + t3
   };
```

$$1 \to \text{CWS}\left[0,\ \frac{3\,\text{CW}[uv]}{2},\ -\frac{13\,\text{CW}[ttu]}{6}+\frac{13\,\text{CW}[tuu]}{6}+\frac{20\,\text{CW}[tuv]}{3}-\frac{23\,\text{CW}[tvu]}{3}-\text{CW}[uuv]+\frac{13\,\text{CW}[uvv]}{3}\right]$$

$$2 \to \text{CWS}\left[-2\,\text{CW}[u],\ 2\,\text{CW}[tu]-3\,\text{CW}[uv],\ -\frac{8\,\text{CW}[ttu]}{3}+3\,\text{CW}[tuu]+\frac{3\,\text{CW}[tuv]}{2}+\frac{2\,\text{CW}[tvu]}{3}-\frac{5\,\text{CW}[uuv]}{2}-\frac{2\,\text{CW}[uvv]}{3}\right]$$

$$3 \to \text{CWS}\left[2\,\text{CW}[u],\ -2\,\text{CW}[tu]+\frac{9\,\text{CW}[uv]}{2},\ \frac{\text{CW}[ttu]}{2}-\frac{5\,\text{CW}[tuu]}{6}+\frac{31\,\text{CW}[tuv]}{6}-\frac{25\,\text{CW}[tvu]}{3}+\frac{3\,\text{CW}[uuv]}{2}+5\,\text{CW}[uvv]\right]$$

$4 \to$ True

4. The meaning(s) of RC

```
Print /@ {
    1 → α,
    2 → (t1 = α // CC[u, γ] // RC[u, -γ]),
    3 → α ≡ t1
   };
```

$1 \to$

$$\text{LS}\left[2\,t-v,\ -\frac{\overline{tu}}{2}+\overline{tv}-2\,\overline{uv},\ -\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{4}{3}\,\overline{t\,\overline{tv}}-\frac{3}{2}\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}-\frac{1}{6}\,\overline{\overline{tu}\,u}-\overline{\overline{tv}\,u}+\overline{\overline{tv}\,v}+2\,\overline{\overline{uv}\,v}\right]$$

$2 \to$

$$\text{LS}\left[2\,t-v,\ -\frac{\overline{tu}}{2}+\overline{tv}-2\,\overline{uv},\ -\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{4}{3}\,\overline{t\,\overline{tv}}-\frac{3}{2}\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}-\frac{1}{6}\,\overline{\overline{tu}\,u}-\overline{\overline{tv}\,u}+\overline{\overline{tv}\,v}+2\,\overline{\overline{uv}\,v}\right]$$

$3 \to$ True

```
Print /@ {
    1 → α,
    2 → (t1 = α // CC[u, γ // RC[u, γ]]),
    3 → (t2 = α // RC[u, γ]),
    4 → t1 ≡ t2
   };
```

$1 \to$

$$\text{LS}\left[2\,t-v,\ -\frac{\overline{tu}}{2}+\overline{tv}-2\,\overline{uv},\ -\frac{5}{3}\,\overline{t\,\overline{tu}}+\frac{4}{3}\,\overline{t\,\overline{tv}}-\frac{3}{2}\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}-\frac{1}{6}\,\overline{\overline{tu}\,u}-\overline{\overline{tv}\,u}+\overline{\overline{tv}\,v}+2\,\overline{\overline{uv}\,v}\right]$$

$$2 \to \text{LS}\left[2\,t-v,\ -\frac{\overline{tu}}{2}+\overline{tv}-2\,\overline{uv},\ -\frac{13}{6}\,\overline{t\,\overline{tu}}+\frac{4}{3}\,\overline{t\,\overline{tv}}-3\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}-\frac{1}{6}\,\overline{\overline{tu}\,u}-3\,\overline{\overline{tv}\,u}+\overline{\overline{tv}\,v}+4\,\overline{\overline{uv}\,v}\right]$$

$$3 \to \text{LS}\left[2\,t-v,\ -\frac{\overline{tu}}{2}+\overline{tv}-2\,\overline{uv},\ -\frac{13}{6}\,\overline{t\,\overline{tu}}+\frac{4}{3}\,\overline{t\,\overline{tv}}-3\,\overline{t\,\overline{uv}}+\frac{1}{3}\,\overline{u\,\overline{uv}}-\frac{1}{6}\,\overline{\overline{tu}\,u}-3\,\overline{\overline{tv}\,u}+\overline{\overline{tv}\,v}+4\,\overline{\overline{uv}\,v}\right]$$

$4 \to$ True

5.

6.

7.

8.

9.

10. The cocycle condition for div

```
Print /@ {
    0 → {α, β},
    1 → (t1 = div[u, α] // ad[u, β]),
    2 → (t2 = div[u, β] // ad[u, α]),
    3 → (t3 = div[u, b[α, β]]),
    4 → (t4 = div[u, α // ad[u, β]]),
    5 → (t5 = div[u, β // ad[u, α]]),
    6 → t1 - t2 ≡ t3 + t4 - t5
   };
```

$$0 \to \left\{\text{LS}\left[-2\,t-2\,u+2\,v,\ -\frac{\overline{t\,v}}{2}+\overline{u\,v},\right.\right.$$
$$\left.-\frac{4}{3}\,\overline{t\,\overline{t\,u}}-\frac{3}{2}\,\overline{t\,\overline{t\,v}}+\frac{3}{2}\,\overline{t\,\overline{u\,v}}+\frac{1}{6}\,\overline{u\,\overline{u\,v}}-\frac{5}{3}\,\overline{\overline{t\,u}\,u}-\frac{2}{3}\,\overline{\overline{t\,v}\,u}+\frac{5}{6}\,\overline{\overline{t\,v}\,v}+\frac{5}{3}\,\overline{\overline{u\,v}\,v}\right],\ \text{LS}\left[2\,u+2\,v,\right.$$
$$\left.\left.2\,\overline{t\,u}-\overline{t\,v}+\frac{3\,\overline{u\,v}}{2},\ \frac{1}{2}\,\overline{t\,\overline{t\,u}}+\frac{5}{3}\,\overline{t\,\overline{t\,v}}+\frac{1}{6}\,\overline{t\,\overline{u\,v}}+\frac{2}{3}\,\overline{u\,\overline{u\,v}}-\frac{7}{6}\,\overline{\overline{t\,u}\,u}-\frac{7}{6}\,\overline{\overline{t\,v}\,u}-2\,\overline{\overline{t\,v}\,v}+\frac{1}{6}\,\overline{\overline{u\,v}\,v}\right]\right\}$$

```
1 → CWS[0, 0, 0]

2 → CWS[0, 0, -CW[tuv] + CW[tvu]]

3 → CWS[0, -4 CW[tu] + 8 CW[uv], -4 CW[ttu] - 4 CW[tuu] + 7 CW[tuv] + 5 CW[uuv] - CW[uvv]]

4 → CWS[0, -4 CW[uv], 4 CW[tuu] - 2 CW[tuv] + 2 CW[tvu] - 3 CW[uuv] - 2 CW[uvv]]

5 → CWS[0, -4 CW[tu] + 4 CW[uv], -4 CW[ttu] + 4 CW[tuv] + 3 CW[tvu] + 2 CW[uuv] - 3 CW[uvv]]

6 → True
```

11.

12.

13.

14.

15.

16.

17.

18. The differential of BCH

```
Print /@ {
    1 → (bch = BCH[u, v]),
    2 → (BCH[u + ϵ t, v + ϵ w] - bch)/ϵ,
    3 → (t1 = (BCH[u + ϵ t, v + ϵ w] - bch)/ϵ // adSeries[(1 - e^-ad)/ad, bch]),
    4 → (t2 = t // adSeries[(1 - e^-ad)/ad, u] // Ad[-v]),
    5 → (t3 = w // adSeries[(1 - e^-ad)/ad, v])
  };
t1 ≡ t2 + t3
```

$$1 \to \text{LS}\left[u+v,\ \frac{\overline{uv}}{2},\ \frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{\overline{uv}v}\right]$$

$$2 \to \text{LS}\left[t+w,\ \frac{\overline{tv}}{2}+\frac{\overline{uw}}{2},\ \frac{1}{12}\overline{t\overline{uv}}+\frac{1}{12}\overline{u\overline{uw}}+\frac{1}{12}\overline{u\overline{vw}}-\frac{1}{12}\overline{\overline{tv}u}+\frac{1}{12}\overline{\overline{tv}v}+\frac{1}{6}\overline{\overline{uw}v}\right]$$

$$3 \to \text{LS}\left[t+w,\ \frac{\overline{tu}}{2}+\overline{tv}-\frac{\overline{vw}}{2},\ \frac{1}{2}\overline{t\overline{uv}}+\frac{1}{6}\overline{v\overline{vw}}+\frac{1}{6}\overline{\overline{tu}u}+\frac{1}{2}\overline{\overline{tv}u}+\frac{1}{2}\overline{\overline{tv}v}\right]$$

$$4 \to \text{LS}\left[t,\ \frac{\overline{tu}}{2}+\overline{tv},\ \frac{1}{2}\overline{t\overline{uv}}+\frac{1}{6}\overline{\overline{tu}u}+\frac{1}{2}\overline{\overline{tv}u}+\frac{1}{2}\overline{\overline{tv}v}\right]$$

$$5 \to \text{LS}\left[w,\ -\frac{\overline{vw}}{2},\ \frac{1}{6}\overline{v\overline{vw}}\right]$$

```
True
```

19. The differential of C

```
Print /@ {
    0 → {α, β, γ},
    1 → (t1 = ((γ // CC[u, α + ϵ β]) - (γ // CC[u, α]))/ϵ),
    2 → (t2 = γ // ad[u, adSeries[(e^ad - 1)/ad, α][β] // RC[u, -α]] // CC[u, α]),
    t1 ≡ t2
  };
```

$$0 \to \left\{ LS\left[-2\,t - 2\,u + 2\,v,\ -\frac{\overline{t\,v}}{2} + \overline{u\,v},\right.\right.$$
$$\left. -\frac{4}{3}\,\overline{t\,\overline{t\,u}} - \frac{3}{2}\,\overline{t\,\overline{t\,v}} + \frac{3}{2}\,\overline{t\,\overline{u\,v}} + \frac{1}{6}\,\overline{u\,\overline{u\,v}} - \frac{5}{3}\,\overline{t\,\overline{u\,u}} - \frac{2}{3}\,\overline{t\,\overline{v\,u}} + \frac{5}{6}\,\overline{t\,\overline{v\,v}} + \frac{5}{3}\,\overline{u\,\overline{v\,v}} \right],$$
$$LS\left[2\,u + 2\,v,\ 2\,\overline{t\,u} - \overline{t\,v} + \frac{3\,\overline{u\,v}}{2},\ \frac{1}{2}\,\overline{t\,\overline{t\,u}} + \frac{5}{3}\,\overline{t\,\overline{t\,v}} + \frac{1}{6}\,\overline{t\,\overline{u\,v}} + \frac{2}{3}\,\overline{u\,\overline{u\,v}} - \right.$$
$$\left. \frac{7}{6}\,\overline{\overline{t\,u}\,u} - \frac{7}{6}\,\overline{\overline{t\,v}\,u} - 2\,\overline{\overline{t\,v}\,v} + \frac{1}{6}\,\overline{\overline{u\,v}\,v} \right], LS\left[t + u - v,\ 2\,\overline{t\,u} + \overline{t\,v} - \frac{3\,\overline{u\,v}}{2},\right.$$
$$\left. \left. -\frac{11}{6}\,\overline{t\,\overline{t\,u}} - \frac{1}{3}\,\overline{t\,\overline{t\,v}} - \frac{5}{6}\,\overline{t\,\overline{u\,v}} + 2\,\overline{u\,\overline{u\,v}} + \frac{7}{6}\,\overline{t\,\overline{u\,u}} - \frac{5}{6}\,\overline{t\,\overline{v\,u}} - \frac{11}{6}\,\overline{t\,\overline{v\,v}} - \frac{11}{6}\,\overline{u\,\overline{v\,v}} \right] \right\}$$

$$1 \to LS\left[0,\ -2\,\overline{u\,v},\ -\frac{3}{2}\,\overline{u\,\overline{u\,v}} + 4\,\overline{\overline{t\,u}\,u} + \overline{\overline{t\,v}\,u} + 7\,\overline{\overline{u\,v}\,v} \right]$$

$$2 \to LS\left[0,\ -2\,\overline{u\,v},\ -\frac{3}{2}\,\overline{u\,\overline{u\,v}} + 4\,\overline{\overline{t\,u}\,u} + \overline{\overline{t\,v}\,u} + 7\,\overline{\overline{u\,v}\,v} \right]$$

True

20. The differential of RC

```
Print /@ {
    0 → {α, β, γ},
    1 → (t1 = ((γ // RC[u, α + ϵ β]) - (γ // RC[u, α]))/ϵ),
    2 → (t2 = γ // RC[u, α] // ad[u, adSeries[(1 - e^-ad)/ad, α][β] // RC[u, α]]),
    t1 ≡ t2
  };
```

$$0 \to \left\{ LS\left[-2\,t - 2\,u + 2\,v,\ -\frac{\overline{t\,v}}{2} + \overline{u\,v},\right.\right.$$
$$\left. -\frac{4}{3}\,\overline{t\,\overline{t\,u}} - \frac{3}{2}\,\overline{t\,\overline{t\,v}} + \frac{3}{2}\,\overline{t\,\overline{u\,v}} + \frac{1}{6}\,\overline{u\,\overline{u\,v}} - \frac{5}{3}\,\overline{t\,\overline{u\,u}} - \frac{2}{3}\,\overline{t\,\overline{v\,u}} + \frac{5}{6}\,\overline{t\,\overline{v\,v}} + \frac{5}{3}\,\overline{u\,\overline{v\,v}} \right],$$
$$LS\left[2\,u + 2\,v,\ 2\,\overline{t\,u} - \overline{t\,v} + \frac{3\,\overline{u\,v}}{2},\ \frac{1}{2}\,\overline{t\,\overline{t\,u}} + \frac{5}{3}\,\overline{t\,\overline{t\,v}} + \frac{1}{6}\,\overline{t\,\overline{u\,v}} + \frac{2}{3}\,\overline{u\,\overline{u\,v}} - \right.$$
$$\left. \frac{7}{6}\,\overline{\overline{t\,u}\,u} - \frac{7}{6}\,\overline{\overline{t\,v}\,u} - 2\,\overline{\overline{t\,v}\,v} + \frac{1}{6}\,\overline{\overline{u\,v}\,v} \right], LS\left[t + u - v,\ 2\,\overline{t\,u} + \overline{t\,v} - \frac{3\,\overline{u\,v}}{2},\right.$$
$$\left. \left. -\frac{11}{6}\,\overline{t\,\overline{t\,u}} - \frac{1}{3}\,\overline{t\,\overline{t\,v}} - \frac{5}{6}\,\overline{t\,\overline{u\,v}} + 2\,\overline{u\,\overline{u\,v}} + \frac{7}{6}\,\overline{t\,\overline{u\,u}} - \frac{5}{6}\,\overline{t\,\overline{v\,u}} - \frac{11}{6}\,\overline{t\,\overline{v\,v}} - \frac{11}{6}\,\overline{u\,\overline{v\,v}} \right] \right\}$$

$$1 \to LS\left[0,\ -2\,\overline{u\,v},\ -\frac{3}{2}\,\overline{u\,\overline{u\,v}} + \overline{\overline{t\,v}\,u} + 7\,\overline{\overline{u\,v}\,v} \right]$$

$$2 \to LS\left[0,\ -2\,\overline{u\,v},\ -\frac{3}{2}\,\overline{u\,\overline{u\,v}} + \overline{\overline{t\,v}\,u} + 7\,\overline{\overline{u\,v}\,v} \right]$$

True

21. The differential of J

```
Print /@ {
    0 → {α, β},
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = ∫_0^1 (div[u, β // RC[u, s α]] // CC[u, -s α]) ds),
    3 → (t2 = ∫_0^1 (s div[u, α // RC[u, s α] //
                ad[u, β // adSeries[(1 - e^(-s ad))/(s ad), α] // RC[u, s α]]] // CC[u, -s α]) ds),
    4 → (t3 = ∫_0^1 (s div[u, α // RC[u, s α]] // ad[u, β // adSeries[(1 - e^(-s ad))/(s ad), α] //
                RC[u, s α]] // CC[u, -s α]) ds),
    t0 ≡ t1 + t2 - t3
  };
```

$$0 \to \left\{\mathrm{LS}\left[-2\,t-2\,u+2\,v,\ -\frac{\overline{t\,v}}{2}+\overline{u\,v},\right.\right.$$
$$\left.-\frac{4}{3}\,\overline{t\,\overline{t\,u}}-\frac{3}{2}\,\overline{t\,\overline{t\,v}}+\frac{3}{2}\,\overline{t\,\overline{u\,v}}+\frac{1}{6}\,\overline{u\,\overline{u\,v}}-\frac{5}{3}\,\overline{t\,\overline{u\,u}}-\frac{2}{3}\,\overline{t\,\overline{v\,u}}+\frac{5}{6}\,\overline{t\,\overline{v\,v}}+\frac{5}{3}\,\overline{u\,\overline{v\,v}}\right],\ \mathrm{LS}\left[2\,u+2\,v,\right.$$
$$\left.\left.2\,\overline{t\,u}-\overline{t\,v}+\frac{3\,\overline{u\,v}}{2},\ \frac{1}{2}\,\overline{t\,\overline{t\,u}}+\frac{5}{3}\,\overline{t\,\overline{t\,v}}+\frac{1}{6}\,\overline{t\,\overline{u\,v}}+\frac{2}{3}\,\overline{u\,\overline{u\,v}}-\frac{7}{6}\,\overline{t\,\overline{u\,u}}-\frac{7}{6}\,\overline{t\,\overline{v\,u}}-2\,\overline{t\,\overline{v\,v}}+\frac{1}{6}\,\overline{u\,\overline{v\,v}}\right]\right\}$$

$$1 \to \mathrm{CWS}\left[2\,\mathrm{CW}[u],\ -\frac{3\,\mathrm{CW}[uv]}{2},\ -\frac{\mathrm{CW}[ttu]}{6}+\frac{\mathrm{CW}[tuu]}{2}+\frac{8\,\mathrm{CW}[tuv]}{3}+\frac{2\,\mathrm{CW}[tvu]}{3}+\frac{\mathrm{CW}[uuv]}{6}-\frac{11\,\mathrm{CW}[uvv]}{3}\right]$$

$$2 \to \mathrm{CWS}\left[2\,\mathrm{CW}[u],\ \frac{\mathrm{CW}[uv]}{2},\ -\frac{\mathrm{CW}[ttu]}{6}-\frac{\mathrm{CW}[tuu]}{6}+\frac{7\,\mathrm{CW}[tuv]}{3}-\frac{5\,\mathrm{CW}[tvu]}{3}+\frac{5\,\mathrm{CW}[uuv]}{3}\right]$$

$$3 \to \mathrm{CWS}\left[0,\ -2\,\mathrm{CW}[uv],\ \frac{2\,\mathrm{CW}[tuu]}{3}-\frac{7\,\mathrm{CW}[tuv]}{3}+5\,\mathrm{CW}[tvu]-\frac{3\,\mathrm{CW}[uuv]}{2}-\frac{11\,\mathrm{CW}[uvv]}{3}\right]$$

$$4 \to \mathrm{CWS}\left[0,\ 0,\ -\frac{8\,\mathrm{CW}[tuv]}{3}+\frac{8\,\mathrm{CW}[tvu]}{3}\right]$$

True

```
Print /@ {
    0 → {α, β};
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = ∫_0^1 (div[u, β // RC[u, s α]] // CC[u, -s α]) ⅆs),
    3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
            ad[u, α // RC[u, s α]]] // CC[u, -s α]) ⅆs),
    4 → (t3 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
          ad[u, α // RC[u, s α]] // CC[u, -s α]) ⅆs),
    5 → (t4 = ∫_0^1 (div[u, b[α // RC[u, s α], β // adSeries[(1 - e^(-s ad))/ad, α] //
            RC[u, s α]]] // CC[u, -s α]) ⅆs),
    t0 ≡ t1 + t2 - t3 - t4
  };
```

$$1 \to \text{CWS}\left[2\,\text{CW}[u],\ -\frac{3\,\text{CW}[uv]}{2},\ -\frac{\text{CW}[ttu]}{6}+\frac{\text{CW}[tuu]}{2}+\frac{8\,\text{CW}[tuv]}{3}+\frac{2\,\text{CW}[tvu]}{3}+\frac{\text{CW}[uuv]}{6}-\frac{11\,\text{CW}[uvv]}{3}\right]$$

$$2 \to \text{CWS}\left[2\,\text{CW}[u],\ \frac{\text{CW}[uv]}{2},\ -\frac{\text{CW}[ttu]}{6}-\frac{\text{CW}[tuu]}{6}+\frac{7\,\text{CW}[tuv]}{3}-\frac{5\,\text{CW}[tvu]}{3}+\frac{5\,\text{CW}[uuv]}{3}\right]$$

$$3 \to \text{CWS}\left[0,\ -2\,\text{CW}[tu]+2\,\text{CW}[uv],\ -\frac{2\,\text{CW}[ttu]}{3}-\frac{8\,\text{CW}[tuu]}{3}+2\,\text{CW}[tuv]+\frac{\text{CW}[tvu]}{6}+\frac{11\,\text{CW}[uuv]}{3}-\frac{3\,\text{CW}[uvv]}{2}\right]$$

$$4 \to \text{CWS}\left[0,\ 0,\ \frac{5\,\text{CW}[tuv]}{6}-\frac{5\,\text{CW}[tvu]}{6}\right]$$

$$5 \to \text{CWS}\left[0,\ -2\,\text{CW}[tu]+4\,\text{CW}[uv],\ -\frac{2\,\text{CW}[ttu]}{3}-\frac{10\,\text{CW}[tuu]}{3}+\frac{5\,\text{CW}[tuv]}{6}-\frac{4\,\text{CW}[tvu]}{3}+\frac{31\,\text{CW}[uuv]}{6}+\frac{13\,\text{CW}[uvv]}{6}\right]$$

```
True
```

```
Print /@ {
    0 → {α, β};
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = ∫_0^1 (div[u, β // RC[u, s α]] // CC[u, -s α]) ⅆs),
    3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
              ad[u, α // RC[u, s α]]] // CC[u, -s α]) ⅆs),
    4 → (t3 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
            ad[u, α // RC[u, s α]] // CC[u, -s α]) ⅆs),
    5 → (t4 = ∫_0^1 (div[u, b[α, β // adSeries[(1 - e^(-s ad))/ad, α]]] // RC[u, s α]] //
          CC[u, -s α]) ⅆs),
    t0 ≡ t1 + t2 - t3 - t4
  };
```

$$1 \to \text{CWS}\left[2\,\text{CW}[u],\ -\frac{3\,\text{CW}[uv]}{2},\ -\frac{\text{CW}[ttu]}{6}+\frac{\text{CW}[tuu]}{2}+\frac{8\,\text{CW}[tuv]}{3}+\frac{2\,\text{CW}[tvu]}{3}+\frac{\text{CW}[uuv]}{6}-\frac{11\,\text{CW}[uvv]}{3}\right]$$

$$2 \to \text{CWS}\left[2\,\text{CW}[u],\ \frac{\text{CW}[uv]}{2},\ -\frac{\text{CW}[ttu]}{6}-\frac{\text{CW}[tuu]}{6}+\frac{7\,\text{CW}[tuv]}{3}-\frac{5\,\text{CW}[tvu]}{3}+\frac{5\,\text{CW}[uuv]}{3}\right]$$

$$3 \to \text{CWS}\left[0,\ -2\,\text{CW}[tu]+2\,\text{CW}[uv],\ -\frac{2\,\text{CW}[ttu]}{3}-\frac{8\,\text{CW}[tuu]}{3}+2\,\text{CW}[tuv]+\frac{\text{CW}[tvu]}{6}+\frac{11\,\text{CW}[uuv]}{3}-\frac{3\,\text{CW}[uvv]}{2}\right]$$

$$4 \to \text{CWS}\left[0,\ 0,\ \frac{5\,\text{CW}[tuv]}{6}-\frac{5\,\text{CW}[tvu]}{6}\right]$$

$$5 \to \text{CWS}\left[0,\ -2\,\text{CW}[tu]+4\,\text{CW}[uv],\ -\frac{2\,\text{CW}[ttu]}{3}-\frac{10\,\text{CW}[tuu]}{3}+\frac{5\,\text{CW}[tuv]}{6}-\frac{4\,\text{CW}[tvu]}{3}+\frac{31\,\text{CW}[uuv]}{6}+\frac{13\,\text{CW}[uvv]}{6}\right]$$

```
True
```

```
Print /@ {
    0 → {α, β};
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = ∫_0^1 (div[u, β // RC[u, s α]] // CC[u, -s α]) 𝕕s),
    3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
            ad[u, α // RC[u, s α]]] // CC[u, -s α]) 𝕕s),
    4 → (t3 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
          ad[u, α // RC[u, s α]] // CC[u, -s α]) 𝕕s),
    5 → (t4 = ∫_0^1 (div[u, β // adSeries[1 - e^(-s ad), α] // RC[u, s α]] // CC[u, -s α]) 𝕕s),
    t0 ≡ t1 + t2 - t3 - t4
  };
```

$$1 \to \text{CWS}\left[2\,\text{CW}[u],\ -\frac{3\,\text{CW}[uv]}{2},\ -\frac{\text{CW}[ttu]}{6}+\frac{\text{CW}[tuu]}{2}+\frac{8\,\text{CW}[tuv]}{3}+\frac{2\,\text{CW}[tvu]}{3}+\frac{\text{CW}[uuv]}{6}-\frac{11\,\text{CW}[uvv]}{3}\right]$$

$$2 \to \text{CWS}\left[2\,\text{CW}[u],\ \frac{\text{CW}[uv]}{2},\ -\frac{\text{CW}[ttu]}{6}-\frac{\text{CW}[tuu]}{6}+\frac{7\,\text{CW}[tuv]}{3}-\frac{5\,\text{CW}[tvu]}{3}+\frac{5\,\text{CW}[uuv]}{3}\right]$$

$$3 \to \text{CWS}\left[0,\ -2\,\text{CW}[tu]+2\,\text{CW}[uv],\ -\frac{2\,\text{CW}[ttu]}{3}-\frac{8\,\text{CW}[tuu]}{3}+2\,\text{CW}[tuv]+\frac{\text{CW}[tvu]}{6}+\frac{11\,\text{CW}[uuv]}{3}-\frac{3\,\text{CW}[uvv]}{2}\right]$$

$$4 \to \text{CWS}\left[0,\ 0,\ \frac{5\,\text{CW}[tuv]}{6}-\frac{5\,\text{CW}[tvu]}{6}\right]$$

$$5 \to \text{CWS}\left[0,\ -2\,\text{CW}[tu]+4\,\text{CW}[uv],\ -\frac{2\,\text{CW}[ttu]}{3}-\frac{10\,\text{CW}[tuu]}{3}+\frac{5\,\text{CW}[tuv]}{6}-\frac{4\,\text{CW}[tvu]}{3}+\frac{31\,\text{CW}[uuv]}{6}+\frac{13\,\text{CW}[uvv]}{6}\right]$$

```
True
```

```
Print /@ {
    0 → {α, β};
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
                ad[u, α // RC[u, s α]]] // CC[u, -s α]) ds),
    3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
              ad[u, α // RC[u, s α]] // CC[u, -s α]) ds),
    4 → (t3 = ∫_0^1 (div[u, β // adSeries[e^(-s ad), α] // RC[u, s α]] // CC[u, -s α]) ds),
    t0 ≡ t1 - t2 + t3
  };
```

$$1 \to \text{CWS}\left[2\,\text{CW}[u],\ -\frac{3\,\text{CW}[uv]}{2},\ -\frac{\text{CW}[ttu]}{6}+\frac{\text{CW}[tuu]}{2}+\frac{8\,\text{CW}[tuv]}{3}+\frac{2\,\text{CW}[tvu]}{3}+\frac{\text{CW}[uuv]}{6}-\frac{11\,\text{CW}[uvv]}{3}\right]$$

$$2 \to \text{CWS}\left[0,\ -2\,\text{CW}[tu]+2\,\text{CW}[uv],\ -\frac{2\,\text{CW}[ttu]}{3}-\frac{8\,\text{CW}[tuu]}{3}+2\,\text{CW}[tuv]+\frac{\text{CW}[tvu]}{6}+\frac{11\,\text{CW}[uuv]}{3}-\frac{3\,\text{CW}[uvv]}{2}\right]$$

$$3 \to \text{CWS}\left[0,\ 0,\ \frac{5\,\text{CW}[tuv]}{6}-\frac{5\,\text{CW}[tvu]}{6}\right]$$

$$4 \to \text{CWS}\left[2\,\text{CW}[u],\ 2\,\text{CW}[tu]-\frac{7\,\text{CW}[uv]}{2},\ \frac{\text{CW}[ttu]}{2}+\frac{19\,\text{CW}[tuu]}{6}+\frac{3\,\text{CW}[tuv]}{2}-\frac{\text{CW}[tvu]}{3}-\frac{7\,\text{CW}[uuv]}{2}-\frac{13\,\text{CW}[uvv]}{6}\right]$$

```
True
```

```
Print /@ {
    0 → {α, β};
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = ∫_0^1 (div[u, β // Ad[-s α] // RC[u, s α]] // CC[u, -s α]) ⅆs),
    3 → (t2 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α] //
                ad[u, α // RC[u, s α]]] // CC[u, -s α]) ⅆs),
    4 → (t3 = ∫_0^1 (div[u, β // adSeries[(1 - e^(-s ad))/ad, α] // RC[u, s α]] //
                ad[u, α // RC[u, s α]] // CC[u, -s α]) ⅆs),
    t0 ≡ t1 + t2 - t3
  };
```

$$1 \to \text{CWS}\left[2\,\text{CW}[u],\ -\frac{3\,\text{CW}[uv]}{2},\ -\frac{\text{CW}[ttu]}{6}+\frac{\text{CW}[tuu]}{2}+\frac{8\,\text{CW}[tuv]}{3}+\frac{2\,\text{CW}[tvu]}{3}+\frac{\text{CW}[uuv]}{6}-\frac{11\,\text{CW}[uvv]}{3}\right]$$

$$2 \to \text{CWS}\left[2\,\text{CW}[u],\ 2\,\text{CW}[tu]-\frac{7\,\text{CW}[uv]}{2},\ \frac{\text{CW}[ttu]}{2}+\frac{19\,\text{CW}[tuu]}{6}+\frac{3\,\text{CW}[tuv]}{2}-\frac{\text{CW}[tvu]}{3}-\frac{7\,\text{CW}[uuv]}{2}-\frac{13\,\text{CW}[uvv]}{6}\right]$$

$$3 \to \text{CWS}\left[0,\ -2\,\text{CW}[tu]+2\,\text{CW}[uv],\ -\frac{2\,\text{CW}[ttu]}{3}-\frac{8\,\text{CW}[tuu]}{3}+2\,\text{CW}[tuv]+\frac{\text{CW}[tvu]}{6}+\frac{11\,\text{CW}[uuv]}{3}-\frac{3\,\text{CW}[uvv]}{2}\right]$$

$$4 \to \text{CWS}\left[0,\ 0,\ \frac{5\,\text{CW}[tuv]}{6}-\frac{5\,\text{CW}[tvu]}{6}\right]$$

```
True
```

```
Print /@ {
    0 → {α, β};
    1 → (t0 = (J[u, α + ϵ β] - J[u, α])/ϵ),
    2 → (t1 = div[u, β // adSeries[(1 - e^(-ad))/ad, α] // RC[u, α]] // CC[u, -α]),
    t0 ≡ t1
  };
```

$$1 \to \text{CWS}\left[\text{CW}[u],\ \frac{3\,\text{CW}[tu]}{2} - 3\,\text{CW}[uv],\right.$$
$$\left.\frac{31\,\text{CW}[ttu]}{6} + \frac{5\,\text{CW}[tuu]}{4} + \frac{7\,\text{CW}[tuv]}{12} - \frac{19\,\text{CW}[tvu]}{4} - \frac{41\,\text{CW}[uuv]}{12} - \frac{3\,\text{CW}[uvv]}{2}\right]$$

$$2 \to \text{CWS}\left[\text{CW}[u],\ \frac{3\,\text{CW}[tu]}{2} - 3\,\text{CW}[uv],\right.$$
$$\left.\frac{31\,\text{CW}[ttu]}{6} + \frac{5\,\text{CW}[tuu]}{4} + \frac{7\,\text{CW}[tuv]}{12} - \frac{19\,\text{CW}[tvu]}{4} - \frac{41\,\text{CW}[uuv]}{12} - \frac{3\,\text{CW}[uvv]}{2}\right]$$

True