Sheffield talk preps

January-19-13
7:26 AM

Switch to the KBH labeling convention:

$x\ y\ z$ for heads Blue! ✓

$u\ v\ w$ for tails Red! ✓

$a\ b\ c$ for ungendered Purple! not done

Find a place for the stepping stones image. ✓

Say "reluctant algebraist"? ✗

Meta–Groups, Meta–Bicrossed–Products, and the Alexander Polynomial, 1

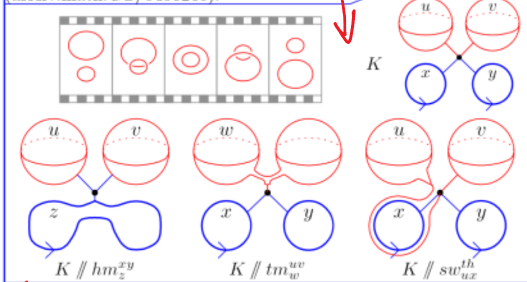
Dror Bar–Natan at the Newton Institute, January 2013.

http://www.math.toronto.edu/~drorbn/Talks/Newton-1301

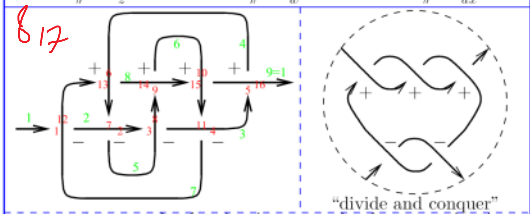
Abstract. I will define "meta-groups" and explain how one specific meta-group, which in itself is a "meta-bicrossed-product", gives rise to an "ultimate Alexander invariant" of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that's a wonderful playground.
This will be a repeat of a talk I gave in Regina in August 2012 and in a number of other places, and I plan to repeat it a good further number of places. Though here at the Newton Institute I plan to make the talk a bit longer, giving me more time to give some further fun examples of meta-structures, and perhaps I will learn from the audience that these meta-structures should really be called something else.

Add "Connected Sum"

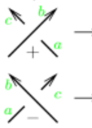
This work is closely related to work by Le Dimet (Comment. Math. Helv. **67** (1992) 306–315), Kirk, Livingston and Wang (arXiv:math/9806035) and Cimasoni and Turaev (arXiv:math.GT/0406269).



8_{17}



A Standard Alexander Formula. Label the arcs 1 through $(n+1)=1$, make an $n\times n$ matrix as below, delete one row and one column, and compute the determinant:



	a	b	c
c	-1	$1-X$	X

	a	b	c
c	$-X$	$X-1$	1

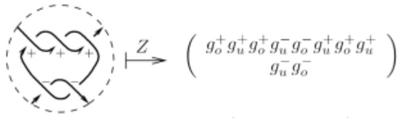
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & X-1 & 0 & -X \\ -1 & X & 0 & 0 & 0 & 0 & 1-X & 0 \\ 0 & -1 & X & 0 & 1-X & 0 & 0 & 0 \\ X-1 & 0 & -X & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-X & 0 & -1 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -X & 1 & 0 & X-1 \\ 0 & 0 & 1-X & 0 & 0 & -1 & X & 0 \\ 0 & 0 & 0 & X-1 & 0 & 0 & -X & 1 \end{pmatrix}$$

```
[[1 ;; 7, 1 ;; 7]] // Det
```

$-1+4X-8X^2+11X^3-8X^4+4X^5-X^6$

Alexander Issues.
- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

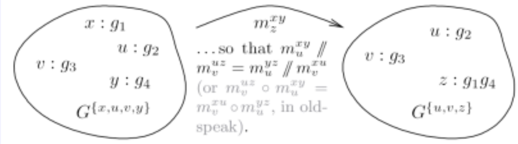
Idea. Given a group G and two "YB" pairs $R^\pm=(g^\pm_o,g^\pm_u)\in G^2$, map them to xings and "multiply along", so that



$$\begin{pmatrix} g^+_o g^+_u g^+_o g^-_u g^-_o g^+_u g^+_o g^+_u \\ g^-_u g^-_o \end{pmatrix}$$

This Fails! R2 implies that $g^\pm_o g^\mp_o=e=g^\pm_u g^\mp_u$ and then R3 implies that g^+_o and g^+_u commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ^z_{xy} for element cloning, ρ^x_y for renamings, and $(D_1,D_2)\mapsto D_1\cup D_2$ for merging, and many obvious composition axioms relating those. $P=\{x:g_1,y:g_2\}\Rightarrow P=\{d_yP\}\cup\{d_xP\}$

A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ, and a collection of operations m^{xy}_z, S_x, e_x, d_x, Δ^z_{xy} (sometimes), ρ^x_y, and $\cup$, satisfying the exact same *linear* properties.

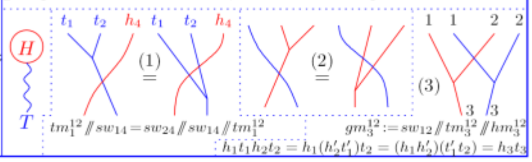
Example 1. The non-meta example, $G_\gamma := G^\gamma$.

Example 2. $G_\gamma := M_{\gamma\times\gamma}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges. Here if $P=\begin{pmatrix} x: & a & b \\ y: & c & d \end{pmatrix}$ then $d_yP=(x:a)$ and $d_xP=(y:d)$ so $\{d_yP\}\cup\{d_xP\}=\begin{pmatrix} x: & a & 0 \\ y: & 0 & d \end{pmatrix}\neq P$. So this G is truly meta.

0

Claim. From a meta-group G and YB elements $R^\pm\in G_2$ we can construct a knot/tangle invariant.

Bicrossed Products. If $G=HT$ is a group presented as a product of two of its subgroups, with $H\cap T=\{e\}$, then also $G=TH$ and G is determined by H, T, and the "swap" map $sw^{th}:(t,h)\mapsto(h',t')$ defined by $th=h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw:T\times H\to H\times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H\times T$, the "bicrossed product".



Meta–Groups, Meta–Bicrossed–Products, and the Alexander Polynomial, 2

A Meta-Bicrossed-Product is a collection of sets $\beta(\eta,\tau)$ and operations tm^{xy}_z, hm^{xy}_z and sw^{th}_{xy} (and lesser ones), such that tm and hm are "associative" and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_\gamma := \beta(\gamma,\gamma)$ and gm as in (3).

Example. Take $\beta(\eta,\tau) = M_{\tau\times\eta}(\mathbb{Z})$ with row operations for the tails, column operations for the heads, and a trivial swap.

β Calculus. Let $\beta(\eta,\tau)$ be

$$\left\{ \begin{array}{c|ccc} \omega & h_1 & h_2 & \cdots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle|\; \begin{array}{l} h_j \in \eta,\ t_i \in \tau, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are rational func-} \\ \text{tions in a variable } X \end{array} \right\},$$

$$tm^{xy}_z : \begin{array}{c|c} \omega & \cdots \\ \hline t_x & \alpha \\ t_y & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_z & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & \eta_1 \\ \hline \tau_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & \eta_2 \\ \hline \tau_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & \eta_1 & \eta_2 \\ \hline \tau_1 & \alpha_1 & 0 \\ \tau_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|cccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$sw^{th}_{xy} : \begin{array}{c|cc} \omega & h_y & \cdots \\ \hline t_x & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_y & \cdots \\ \hline t_x & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$ and $\langle c\rangle := \sum_i c_i$, and let

$$R^p_{xy} := \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & X-1 \\ t_y & 0 & 0 \end{array} \qquad R^m_{xy} := \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & X^{-1}-1 \\ t_y & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On braids, it is equivalent to the Burau representation. A variant for links contains the multivariable Alexander polynomial.

Why Happy? • Applications to w-knots.
• Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribbonness, cabling, v-knots, knotted graphs, etc., and there's potential for vast generalizations.
• The least wasteful "Alexander for tangles" I'm aware of.
• Every step along the computation is the invariant of something.
• Fits on one sheet, including implementation & propaganda.

Further meta-monoids. Π (and variants), $\mathcal{A}$ (and quotients), vT, …

Further meta-bicrossed-products. Π (and variants), $\overrightarrow{\mathcal{A}}$ (and quotients), M_0, M, $\mathcal{K}^{bh}$, $\mathcal{K}^{rbh}$, …

Meta-Lie-algebras. $\mathcal{A}$ (and quotients), $\mathcal{S}$, …

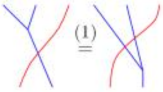
Meta-Lie-bialgebras. $\overrightarrow{\mathcal{A}}$ (and quotients), …

I mean business!

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, A_]] := B[βSimp[ω],
  Collect[A, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, A_]] := Module[{ts, hs, M},
  ts = Union[Cases[B[ω, A], t_x_ :> x, Infinity]];
  hs = Union[Cases[B[ω, A], h_x_ :> x, Infinity]];
  M = Outer[βSimp[Coefficient[A, h_#1 t_#2]] &, hs, ts];
  PrependTo[M, t_# & /@ ts];
  M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
  MatrixForm[M]];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
⟨A_⟩ := A /. t_ -> 1;
tm_{x_,y_→z_}[β_] := βCollect[β /. t_{x|y} -> t_z];
hm_{x_,y_→z_}[B[ω_, A_]] := Module[
  {α = D[A, h_x], β = D[A, h_y], γ = A /. h_{x|y} -> 0},
  B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{x_,y_}[B[ω_, A_]] := Module[{α, β, γ, δ, ε},
  α = Coefficient[A, h_y t_x]; β = D[A, t_x] /. h_y -> 0;
  γ = D[A, h_y] /. t_x -> 0;   δ = A /. h_y | t_x -> 0;
  ε = 1 + α;
  B[ω ε, α (1 + ⟨γ⟩/ε) h_y t_x + β (1 + ⟨γ⟩/ε) t_x
     + γ/ε h_y               + δ - γ β/ε
  ] // βCollect];
gm_{x_,y_→z_}[β_] := β // sw_{xy} // hm_{xy→z} // tm_{xy→z};
B /: B[ω1_, A1_] B[ω2_, A2_] := B[ω1 ω2, A1 + A2];
Rp_{x_,y_} := B[1, (X - 1) t_x h_y];
Rm_{x_,y_} := B[1, (X^-1 - 1) t_x h_y];
```

```
{β = B[ω, Sum[α_{10 i+j} t_i h_j, {i, {1, 2, 3}}, {j, {4, 5}}]],
 (β // tm_{12→1} // sw_{14}) == (β // sw_{24} // sw_{14} // tm_{12→1})}
```

$$\left\{ \begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}, \text{True} \right\}$$



Some testing

```
{Rm_{51} Rm_{62} Rp_{34} // gm_{14→1} // gm_{25→2} // gm_{36→3},
 Rp_{61} Rm_{24} Rm_{35} // gm_{14→1} // gm_{25→2} // gm_{36→3}}
```

$$\left\{ \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix} \right\}$$

… divide and conquer!

```
β = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15}
```

8_{17}

$$\begin{pmatrix} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{-1+X}{X} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X & 0 \\ t_8 & 0 & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X \\ t_{12} & -\frac{-1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+X & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+X & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Do[β = β // gm_{1k→1}, {k, 2, 10}]; β
```

8_{17}, cont.

$$\begin{pmatrix} \frac{1}{X} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+X)(1+X)}{X} & -(-1+X)(1-X+X^2) & (-1+X)(1-X+X^2) & -1+X \\ t_{12} & -\frac{-1+X}{X} & 0 & 0 & 0 \\ t_{14} & -1+X & \frac{(-1+X)^2(1-X+X^2)}{X} & -\frac{(-1+X)^2(1-X+X^2)}{X} & 0 \\ t_{16} & \frac{-1+X}{X} & (-1+X)^2 & -\frac{(-1+X)^3}{X} & 0 \end{pmatrix}$$

James Waddell Alexander

```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

$$\left(-\frac{1-4X+8X^2-11X^3+8X^4-4X^5+X^6}{X^3} \right)$$

A Partial To Do List. 1. Where does it *more simply* come from?
2. Remove all the denominators.
3. How do determinants arise in this context?
4. Understand links. (i.e., meta-conjugacy classes.)
5. Find the "reality condition".
6. Do some "Algebraic Knot Theory".
7. Categorify.
8. Do the same in other natural quotients of the v/w-story.

trivial

ribbon

example

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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