Proving the J properties

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9:16 AM

Continued 2013-03

$$J_u(\lambda_0) := \int_0^1 ds\ \mathrm{div}_u(\lambda_0 /\!\!/ RC_u^{s\lambda_0}) /\!\!/ C_u^{-s\lambda_0}.$$ implies

(1) $$J_u(\mathrm{bch}(\lambda_x,\lambda_y)) = J_u(\lambda_x) + J_u(\lambda_y /\!\!/ RC_u^{\lambda_x}) /\!\!/ C_u^{-\lambda_x},$$

and

$$J_w(\lambda_x /\!\!/ tm_w^{uv}) /\!\!/ RC_w^{\lambda_x /\!\!/ tm_w^{uv}} = J_u(\lambda_x) /\!\!/ RC_u^{\lambda_x} /\!\!/ \overbrace{RC_v^{\lambda_x /\!\!/ RC_u^{\lambda_x}} /\!\!/ tm_w^{uv}}^{= tm_v^{uv} /\!\!/ RC_w^{\lambda_x /\!\!/ tm_w^{uv}}}$$

(2) $$+ J_v(\lambda_x /\!\!/ RC_u^{\lambda_x}) RC_v^{\lambda_x /\!\!/ RC_u^{\lambda_x}} /\!\!/ tm_w^{uv}$$

For (2): We have

$$\mathrm{div}_w(\gamma /\!\!/ tm_w^{uv}) = (\mathrm{div}_u(\gamma) + \mathrm{div}_v(\gamma)) /\!\!/ tm_w^{uv}$$

Now with $\gamma_w := \gamma /\!\!/ tm_w^{uv}$,

$$J_w(\gamma_w) = \int_0^1 ds\ \mathrm{div}_w(\gamma_w /\!\!/ RC_w^{s\gamma_w}) /\!\!/ C_w^{-s\gamma_w}$$

use $$\lambda /\!\!/ (u,v \to w) /\!\!/ RC_w^{\lambda_x /\!\!/ (u,v\to w)} = \lambda /\!\!/ RC_u^{\lambda_x} /\!\!/ RC_v^{\lambda_x /\!\!/ RC_u^{\lambda_x}} /\!\!/ (u,v \to w). \qquad (11)$$

to get

$$J_w(\gamma_w) = \int_0^1 ds\ \mathrm{div}_w \underbrace{\left[\gamma /\!\!/ RC_u^{s\gamma} /\!\!/ RC_v^{s\gamma /\!\!/ RC_u^{s\gamma}} /\!\!/ tm_w^{uv}\right]}_{\text{splits! But I still need to digest the } C^{s\gamma}/RC^{s\gamma} \text{ algebra}} /\!\!/ C_w^{-s\gamma_w}$$

For (1): we have

$$\mathrm{div}_u(\mathrm{bch}(\lambda_x, \lambda_y)) \overset{?}{\sim} \mathrm{div}_u \lambda_x + (\mathrm{div}_u \lambda_y) /\!\!/ C_u^{\lambda_x}$$

$$J_u(\lambda_x) \overset{?}{\sim} \mathrm{div}_u\left(\frac{e^{D_u^{\lambda_x}} - 1}{D_u^{\lambda_x}} \lambda_x\right)$$

The 1-cocycle condition for div:

In tder language:

$$\alpha(\operatorname{div}\beta)-\beta(\operatorname{div}\alpha)=\operatorname{div}([\alpha,\beta])$$

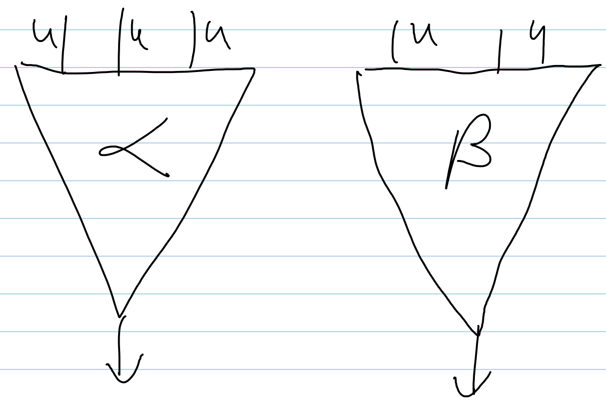
In present language: with $ad_u^\gamma(\lambda):=\lambda /\!/ \operatorname{der}(u\to[\gamma,u])$,

$$(\operatorname{div}_u\beta)/\!/ad_u^\alpha-(\operatorname{div}_u\alpha)/\!/ad_u^\beta \overset{?}{=} \operatorname{div}_u\left([\alpha,\beta]+ad_u^\alpha\beta-ad_u^\beta\alpha\right)$$

more or less

There should also be a condition relating div_u and div_v.

Is the cyclic property of CW used here? Yes, see [The cocycle property for div](#) in 2013-03



$$[\alpha,\beta]_u\sim[\alpha,\beta]+D_u^\alpha\beta-D_u^\beta\alpha \qquad \text{"}tder_u\text{"}$$

Is this a free Lie algebra? No!

[but it is an alt. Lie structure on the same set]

$[u,u]_u=0$

$[v,v]_u=0$

$[u,v]_u=[u,v]+\cancel{D_u^u}v-D_u^v u=$

$=[u,v]\pm[v,u]=?$ should be 0

It would be nice to find a presentation for this Lie algebra

Both FL & $tder_u$ act on FL. Is there a map $FL \longrightarrow tder_u = FL$ that carries on action to the other?

Given $\alpha \in FL$ find $\beta \in FL$ s.t.

$$e^{D_u^\alpha} = e^{der(u \to [\alpha, u])} = C_u^\beta \left(= (u \to e^{ad\,\beta} u)\right)$$

roughly $\beta = \frac{e^{D_u^\alpha} - 1}{D_u^\alpha} \alpha$ or something similar.

How does bch in $tder_u$ compare to bch in FL?

In the AT case, what does j give, graph-theoretically?

$$0 \longrightarrow \text{wheels} \longrightarrow PA^w \rightleftarrows \text{trees} \longrightarrow 0$$

div is the difference between two splittings.

Can I write the bracket of PA_u^w in terms of the bracket of FL and div?

Not the bracket the issue is, but the antipode?

Aside: an antipode is all that is needed for a swap map:

$th \longmapsto (th)^{-1} = h^{-1}t^{-1} \ldots$

If G, H, T have antipodes, then there's a swap map.

Could it be that a crossed homomorphism is the difference between two antipodes?

Perhaps I should study groups of the kind $C \cdot T \cdot H$, where C is central?

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] & G \arrow[r] \arrow[loop above, "S_0"] \arrow[loop below, "S"] & T \arrow[r] & 1 \end{tikzcd}$$

$(th)^{-1} = h^{-1}t^{-1} = h^{-1}t^{-1}hh^{-1}$

$A \to AA^T$ $\qquad$ $AB \to ABB^TA^T = A(BB^T)A^{-1}AA^T$

$$= J(B)^A + J(A)$$

$\cdots >$ a 1-cocycle compares A^{-1} & A^T

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that is $u \mapsto x^T u x$, not $u \mapsto x^{-1}ux$!