Lochak on "A brief introduction to the etale fundamental group and to the Galois action"

February-19-13
5:09 PM

Lochak: 1. Introduction

2. Reminder on group completions.

3. [SGA 1] – Some instructions for use.

4. (Arithmetic) Galois action on $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})$

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1. Introduction

* Grothendieck "Esquise d'un programme" (1984) top.
* Drinfel'd 89 alg. top.
* Ihara 1986 arithmetic.
* Deligne 1989 Alg. geom.

In Grothendieck, GT approximates $G_{\mathbb{Q}}$

$$G_{\mathbb{Q}} \hookrightarrow GT$$

In Drinfel'd $GT(\mathbb{Q})$ appears as the univ. deformation group of braided tensor categories. (pro-unipotent version.)

In Deligne $GT(\mathbb{Q})$ approximates $\pi_1(MT(\mathbb{Z}))$

In fact, $\pi_1(MT(\mathbb{Z})) \subset GT(\mathbb{Q})$

Tate motives.

[SGA 1] $\longrightarrow$ "Esquise" all about π_1!

1960 1984

"a huge generalization

of Galois Theory."

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2. Group Completions

a. $G = \pi_1^{top}(X)$ $\quad X/\mathbb{C}$ quasi-projective alg. variety

discrete & finitely generated.

b. $Gal(E/\mathbb{K})$

Thm $G_{\mathbb{K}}$ is not finitely generated if $\mathbb{K}$ is a number field.

(but it is pro-finite)

Start from G discrete & finitely generated

$$\hat{G} = \varprojlim_{\substack{N \triangleleft G \\ (G:N) < \infty}} G/N = \varprojlim_{\substack{I \text{ invariant under} \\ \text{any automorphism, } (G:I) < \infty}} G/I$$

__Lemma__ Invariant subgroups are cofinal

__Property__ $Aut(\hat{G})$ is pro-finite $=:$ a projective limit of finite groups.

$$G_{\mathbb{Q}} \subset \mathbb{I}\Gamma \subset \widehat{GT} \subset \widehat{GT}_0 \subset Aut(\widehat{F_2})$$

$$F_2 = \mathbb{Z} * \mathbb{Z} = \pi_1^{top}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$$

$Aut\, \mathbb{Z} = \pm 1 \qquad Aut(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^{\times}$

__Thm__ $Out(F_2) = GL_2(\mathbb{Z}) = Aut(F_2^{ab})$

"All" genus G-T group"

$G^{pronil} = \varprojlim G/N$ N - cofinite, G/N Nilpotent

<u>Prop</u> A finite nilpotent group is the direct product of its Sylow subgroups.

So $G^{pronil} = \prod_{l} G^{pro\text{-}l}$ (G/N an l-group)

"Quillen appendix to rational homotopy theory"

$G^{pro\text{-}unit}$ = The group-like elements in $K\widehat{(G)}$ ← complete using augmentation ideal.

"For $\mathbb{Q}$ this is Malcev"

$\exists\ G^{pro\text{-}l} \xrightarrow[\text{if } G \text{ is torsion-free}]{\text{injective}} G(\mathbb{Q}_l)$

<u>Prop</u> G group, T: Torsion elements,

$G^{pro\text{-}uni} \cong (G/T)^{pro\text{-}uni}$

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3. [SGA 1]

1. Definition of etale π_1
2. GAGA Riemann's existence theorem.
3. Galois short exact sequence.

4. The Galois group as an endo-functor.
5. What do we know about the Galois action in general.

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1. Etale π_1: X a scheme or a 1-stack

$$\pi_1(X) = \varprojlim_Y \mathrm{Aut}(Y/X)$$

Y runs over pointed étale galois covers. (finite & connected)

Example take $X = \mathbb{P}^1 \setminus \{0,\infty\} = \mathbb{C}^\times$

$$\left.\begin{array}{cc} Y & u^n = t \\ \downarrow & \\ X & t \end{array}\right\} \mathbb{Z}/n \qquad \pi_1(X) = \varprojlim_n \mathbb{Z}/n = \hat{\mathbb{Z}}$$

$$\pi_1^{top}(X) = \mathbb{Z}.$$

Etale cover is $Y \xrightarrow{f} X$ s.t. f is finite, surjective, unramified, flat.

finite:
$$\begin{array}{c} Y = \mathrm{spec}(B) \\ \downarrow \\ X = \mathrm{spec}(A) \end{array}$$

$A \subset B$
B is a finite A-module.

unramified: B/A unramified

flat:

Thm If X is normal then any unramified

Thm If X is normal then any unramified $Y \to X$ is flat. 51:30

GAGA: $\widehat{\pi_1(X) \cong \pi_1^{top}(X)}$

in [SGA 1] chap XII by trancendental methods 55:39

X — connected $\mathbb{C}$-scheme.

If $\mathbb{K}$ is a field, $\pi_1(\mathrm{Spec}(\mathbb{K})) = G_{\mathbb{K}}$

Extension of alg. closed fields 1:00:00

The Galois action 1:02:40

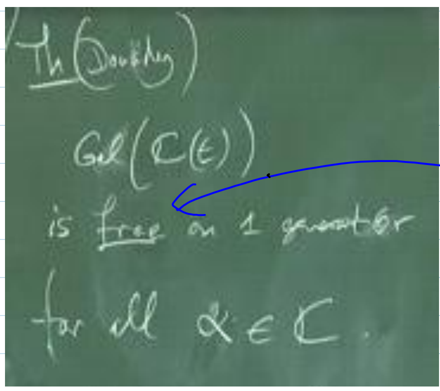
$X/\mathbb{K} \rightsquigarrow X \otimes_{\mathbb{K}} \overline{\mathbb{K}} = \overline{X} \quad \pi_1(\overline{X}) =: \pi_1^{geom}(X)$

(Example $\mathrm{Spec}(\mathbb{Q}(t)/t^2-2)$ is connected but not geometrically connected)

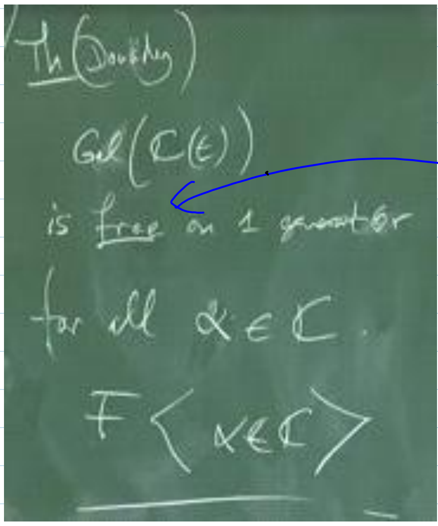
Exact sequence:

$$1 \to \pi_1^{geom}(X) \longrightarrow \pi_1(X) \to G_{\mathbb{K}} \to 1$$

1:36:00



Free profinite.



Free profinite.

1:36:00

1:12:00

On to $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$

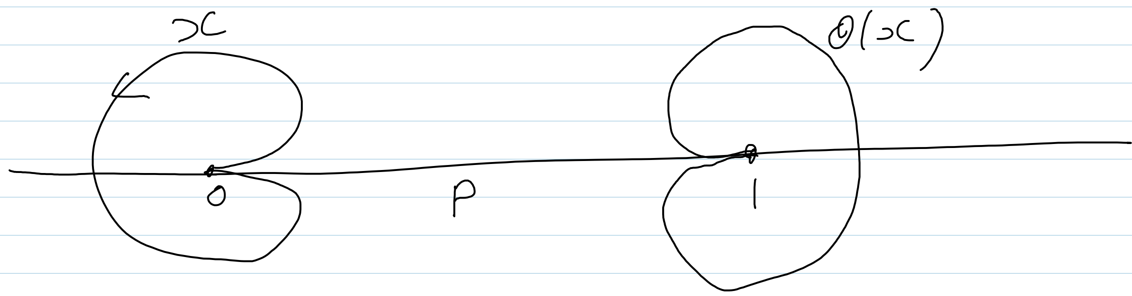
$$\pi_1^{geom}(X) \simeq \pi_1(X_{\overline{\mathbb{Q}}}) \simeq \pi_1(X_{\mathbb{C}}) \simeq \pi_1^{top}(X_{\mathbb{C}}^{an}) \simeq \widehat{F_2}$$

So

$$1 \to \widehat{F_2} \to \pi_1(X) \to G_{\mathbb{Q}} \to 1$$

So

$$G_{\mathbb{Q}} \longrightarrow \mathrm{Out}(\widehat{F_2})$$



The last few minutes are about how the cyclotomic character arises.