John Jones on "The Hochschild cochain complex, operads, and the Deligne conjecture I"

February-05-13
4:34 AM

http://www.newton.ac.uk/programmes/GDO/seminars/2013020510301.html

If the proj way will ever work, this will become so much prettier.

Gerstenhaber Algebras

A G_n-algebra is a graded v.s. w/ two ops:

1. A degree 0 product $V_p \otimes V_q \to V_{p+q}$, graded commutative and associative.

2. A degree n product $[\,,\,]$

$$V_p \otimes V_q \longrightarrow V_{p+q+n}$$

with $[a,b] = (-1)^{(|a|-n)(|b|-n)}[b,a]$

For the relations, introduce $L_a(x) := [a,x]$
$\deg L_a = |a|+n$

Relations L_a is a graded derivation of degree $|a|+n$ of both $[\,,\,]$ and of $\cdot$ product.

The Hochschild Complex V: v.s.

$C^n(V,V) = \mathrm{Hom}(V^{\otimes n}, V)$ an operad

$\circ_i$: $f \in C^n(V,V)$, $g \in C^m(V,V)$ gives

$f \circ_i g \in C^{n+m-1}(V,V)$ by

$f \circ (1^{\otimes(i-1)} \otimes g \otimes 1^{\otimes(n-i)})$

Quote: "We work mod 2 over the ...

There's also ...

There's also

$$f \circ g = \sum (-1)^i f \circ_i g$$

mod 2 over the rationals"

$$[f,g] = f \circ g \pm g \circ f$$

Lemma $\mu \in C^2(V,V)$ is associative iff $\mu \circ \mu = 0$

Now take $V = A$, an algebra w/ product μ

This gives a "cup product"

$$\cup : C^p(A,A) \otimes C^q(A,A) \longrightarrow C^{p+q}(A,A)$$
$$f \otimes g \longmapsto \mu \circ (f \otimes g)$$

associative but not graded commutative.

There's also a differential

$$d_\mu(f) = [\mu, f]$$

$$\mu \circ \mu = 0 \implies d_\mu \circ d_\mu = 0$$

In fact d_μ is the usual Hochschild coboundary operator b.

$$HH^*(A,A) := H^*(C^\bullet(A,A), d_\mu)$$

Braces: $f, g_1, \ldots g_k \in C^\bullet(A,A)$

$$f\{g_1 \ldots g_k\} = \sum_I f \circ (1^{\otimes i_1} \otimes g_1 \otimes 1^{\otimes i_2} \otimes g_2 \ldots\ldots)$$

If $|f|<k$ this is 0

If $|f|=k$ this is a sum w/ one term

⋮

Famous Formula:

$$(f\circ g)\circ h - f\circ(g\circ h) = f\{g,h\} + (-1)^{|g||h|} f\{h,g\}$$

So the cup product is commutative up to chain homotopy:

$$f\cup g - (-1)^{|f||g|} g\cup f = d_\mu(f\circ g) \pm (d_\mu f)\circ g \pm f\circ d_\mu g$$

So HH has a graded commutative product; it also has $[\,,\,] : HH^p \otimes HH^q \to HH^{p+q-1}$, so

<u>Thm</u> (Gerstenhaber) $HH^*(A,A)$ is a e_{-1}-algebra.

Why care?

A associative algebra. A $*$-product on A is a $\Bbbk[[t]]$ bilinear associative product on $A[[t]]$, s.t.

$$a * b = ab \pmod t$$

"A formal deformation of $\cdot$"

A $*$ product is determined by

$$\theta : A\otimes A \longrightarrow A[[t]]$$

w/ $\Theta(a,b) = \mu(a,b) + C(a,b)$ $\quad C \equiv 0 \bmod t$

Associative $\Leftrightarrow \Theta \circ \Theta = 0 \Leftrightarrow [\Theta, \Theta] = 0$

$\Leftrightarrow [\mu + C, \mu + C] = 0$

$\Leftrightarrow 2[\mu, C] + [C, C] = 0$

$\Leftrightarrow d_\mu C + \frac{1}{2}[C, C] = 0$ "the MC eqn"

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d-fold loop spaces. (Y, y_0) top space w/ base

$\Omega^d(Y) := Map((D^d, S^{d-1}), (Y, y_0))$

$= \Omega(\Omega^{d-1}(Y))$

Concentrate on $d=2$: $X = \Omega^2 Y$

$\mu : X \times X \longrightarrow X$ is homotopy commutative

Now define operations on $S_*(X)$

(sing. chains): (pontriagin prod.)

$* : S_p(X) \otimes S_q(X) \longrightarrow S_{p \times q}(X \times X) \longrightarrow S_{p+q}(X)$

descends to homology...

Choose a homotopy $F : I \times X \times X \longrightarrow X$ s.t.

$F(0, x, y) = \mu(x, y) \quad F(1, x, y) = \mu(y, x)$

This gives an operation

$\lambda : S_p(X) \otimes S_q(X) \longrightarrow S_{p+q+1}(X)$

$$\partial\lambda(a,b) \pm \lambda(\partial a, b) \pm \lambda(a, \partial b) =$$
$$= a * b - (-1)^{|a||b|}\, b * a$$

... So $H_*(X)$ is g-commutative

So $H_*(X)$ is a g_{+1}-algebra.

Thm (Fred Cohen) $H_*(\Omega^2\Sigma^2 X; \mathbb{Q})$ is a free G_1-algebra generated by $H_*(X)$

Side remark $\Omega^2\Sigma^2 X$ is the "free double loop space generated by X" .. see video

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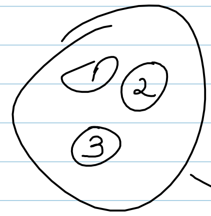
Deligne Conjecture (vague version): The Hochschild complex $C^*(A,A)$ should have the same algebraic structure as $S_*(\Omega^2 X)$

We'll now try to make this precise...

The little discs operad:

$D_n(k)$ space of k ordered disjoint lin. embedded n-discs in D^n: $\left\{ \right.$  $\left. \right\}$

$\exists$ centre map: $D_n(k) \longrightarrow C_k(\mathbb{R}^n)$:= conf. space in $\mathbb{R}^n$

this is a homotopy equiv.

If $n \geq 3$, $\pi_1(D_n(k)) = \pi_1(C_k(\mathbb{R}^n)) = 1$

$n=2$ $\pi_1(D_n(k)) = PB_k$

I'm not convinced that it isn't the case that homotopy theorists love their machine, hence they use it even if it is not necessary.

$n=2$ $\pi_j(D_n(k)) = 0$ if $j > 1$

We know $H^*(C_k(\mathbb{R}^n))$: $\exists$ maps:

$P_{ij}: C_k(\mathbb{R}^n) \longrightarrow S^{n-1}$ $\quad x_i, x_j \mapsto \frac{x_i - x_j}{|\cdot|}$

so get $a_{ij} \in H^{n-1}(C_{k-1}(\mathbb{R}^n))$

by pulling back the generator of $H^*(S^{n-1})$

Then $H^*(C_k(\mathbb{R}^n))$ is the ring generated by a_{ij}'s mod:

1. $a_{ij} = (-1)^n a_{ji}$

2. $a_{ij} a_{jk} + a_{jk} a_{ki} + a_{ki} a_{ij} = 0$

3 if n is odd, $a_{ij}^2 = 0$

Top dim cohomology

$$L_n(k) = H^{(n-1)(k-1)}\big(C_k(\mathbb{R}^n)\big) \quad \dim = (k-1)!$$

is a representation of S_k; it is the $(1,\dots,1)$ part of $FL(x_1,\dots,x_k)$ (graded Lie, $\deg x_i = n-1$)

... related to the Lie operad.

D_n is an operad in the category of topological spaces, in the usual way.

① $\Omega^2 Y$ is a D_2-space.

② If Z is a connected D_2 space then there is a space Y s.t.

$$Z = \Omega^2 Y$$

$$\begin{array}{ll} \text{Topological operad} & D_2 = \{D_2(k)\}_{k \geq 1} \\ \Downarrow & \\ \text{dg-operad } S_*(D_n) & \\ \Downarrow \text{homology} & \\ \text{graded operad} & \{H_*(D_2(k))\} \end{array}$$

$S_*(\Omega^2 X)$ is an $S_*(D_n)$-algebra

<u>Deligne Conjecture</u>: (mark 1)

$C^*(A,A)$ is an algebra over the operad

$$S_*(D_2)$$

(The non-triviallity condition was mentioned in the following lecture)