

Deformation Quantization

$$\mathcal{O}_V = C^\infty(\mathbb{R}^n) \text{ or } S(V^*) \text{ or } \hat{S}(V^*)$$

Assume $\{, \}$ a poisson bracket on \mathcal{O}_V
want an associative $\mathbb{K}[[\hbar]]$ -linear product on $\mathcal{O}_V[[\hbar]]$ st.

1. $F * G = FG + O(\hbar)$

2. $\frac{1}{2}(F * G - G * F) = \hbar \{F, G\} + O(\hbar^2)$

Why should we care?

we care?

What will it buy us?

This is "a quantization".

Thm (Kontsevich) one can do this.

Strategy: Re-interpret as solving MCE in some dg-Lie algebra

* Poisson brackets are sol'n of MCE in

$$\text{Der}(\mathcal{O}_V)[-1]$$

(meaning that the derivations are in degree 1)



$$S_{\mathcal{O}_V}(\text{Der}(\mathcal{O}_V)[-1])[1]$$

w/ 0 differential

$$\parallel \\ T_{\text{poly}} V$$

* Associative deformations of the product on an algebra $A \iff$ MCE in

$$C^0(A, A)[\hbar] \quad d = d_H, \quad [\] = [\]_G$$

$$m + \mu \text{ associative} \iff d\mu + \frac{1}{2}[\mu, \mu] = 0$$

Remarks on MCE:

• If $\psi: g \rightarrow h$ is a morphism of dg-Lie algebras, then

$$\psi(MC(g)) \subset MC(h)$$

Better, gauge equivalences are sent to gauge equivalences

$x \in \hbar g^0[[\hbar]]$ is pro-nilpotent

acts on $\hbar g^1[[\hbar]] \ni \alpha$ by

$$x \cdot \alpha = [x, \alpha] + d(x)$$

$\text{Gauge}(g) = \exp(\hbar g^0[[\hbar]])$ acts on $\hbar g^1[[\hbar]]$

this action preserves MC elements.

$$\overline{MC}(g) = MC(\hbar g^1[[\hbar]]) / \text{Gauge}(g)$$

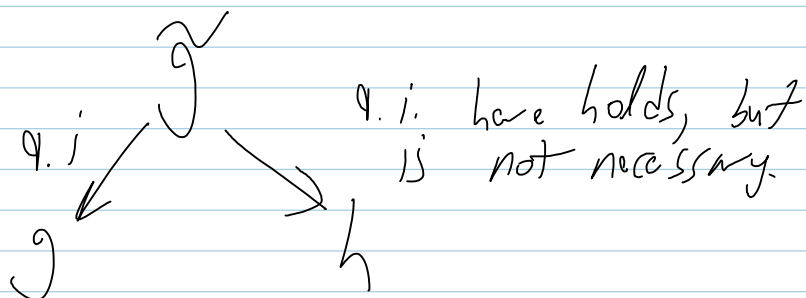
Any morphism $\psi: g \rightarrow h$ induces

$$\overline{MC}(g) \rightarrow \overline{MC}(h)$$

Thm If ψ is quasi-iso, then this is a bijection.

$$g = \text{Tray } V \quad h = C^\circ(\mathcal{O}_V, \mathcal{O}_V)[1]$$

we need



Bruno: One can replace zig-zags by ∞ -morphisms

$$g \rightsquigarrow h$$

Thm (Konts.) $\exists \infty$ - ψ -iso of dg-Lie algs 18:40

$$\psi: \text{Tray } V \rightsquigarrow C(\mathcal{O}_V, \mathcal{O}_V)[1]$$

The explicit map

$$\overline{MC}(\text{Tray } V) \longrightarrow \overline{MC}(C(\mathcal{O}_V, \mathcal{O}_V)[1])$$

$$\downarrow \alpha \longrightarrow \sum_{|K| \geq 1} \frac{1}{|K|!} \psi^K(\alpha \dots \alpha)$$

Recall The definition of ∞ -morphisms of dg-Lie algebras, L_∞ -morphisms.

it is a morphism of co-commutative dg-

coalg

22:30

$$S^c(g[\Sigma]) \longrightarrow S^c(h[\Sigma])$$