Calaque: Formality, obstructions to formality, and a key argument in the proof of the formality conjecture I

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Deformation Quantization

$\mathcal{O}_V = C^\infty(\mathbb{R}^n)$ or $S(V^*)$ or $\hat{S}(V^*)$

Assume $\{,\}$ a poisson bracket on $\mathcal{O}_V$
want an associative $\mathbb{K}[[\hbar]]$-linear product on $\mathcal{O}_V[[\hbar]]$ s.t.

1. $f * g = fg + O(\hbar)$
2. $\frac{1}{2}(f*g - g*f) = \hbar\{f,g\} + O(\hbar^2)$

Why should we care?

This is "a quantization".

What will it buy us?

Thm (Kontsevich) One can do this.

Strategy: Re-interpret as solving MCE in some dg-Lie algebra

* Poisson brackets are sol'n of MCE in

$\mathrm{Der}(\mathcal{O}_V)[-1]$ (meaning that the derivations are in degree 1)

$\downarrow$

$S_{\mathcal{O}_V}(\mathrm{Der}(\mathcal{O}_V)[-1])[1]$ w/ 0 differential

$\|$

$T_{poly} V$

* Associative deformations of the product on an algebra $A \longleftrightarrow$ MCE in

$$C^{\bullet}(A,A)[1] \qquad d = d_H, \; [\;] = [\;]_G$$

$$m + \mu \text{ associative} \Longleftrightarrow d\mu + \tfrac{1}{2}[\mu,\mu] = 0$$

Remarks on MCE:

- If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a morphism of dg-Lie algs, then

$$\varphi(MC(\mathfrak{g})) \subset MC(\mathfrak{h})$$

Better, gauge equivalences are sent to gauge equivalences

$x \ni \hbar\mathfrak{g}^0[[\hbar]]$ is pro-nilpotent

acts on $\hbar\mathfrak{g}^1[[\hbar]] \ni \alpha$ by

$$x \cdot \alpha = [x, \alpha] + d(x)$$

$Gauge(\mathfrak{g}) = \exp(\hbar\mathfrak{g}^0[[\hbar]])$ acts on $\hbar\mathfrak{g}^1[[\hbar]]$

this action preserves MC elements.

$$\overline{MC}(\mathfrak{g}) = MC(\hbar\mathfrak{g}^1[[\hbar]]) / Gauge(\mathfrak{g})$$

Any morphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ defines

$$\overline{MC}(\mathfrak{g}) \longrightarrow \overline{MC}(\mathfrak{h})$$

Thm If φ is quasi-iso, then this is a bijection.

$$\mathfrak{g} = T_{poly} V \qquad \mathfrak{h} = C^{\cdot}(\mathcal{O}_V, \mathcal{O}_V)[1]$$

We need

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "q.i"'] \arrow[dr] & \\ \mathfrak{g} & & \mathfrak{h} \end{tikzcd}$$

q.i. here holds, but is not necessary.

Bruno: One can replace zig-zags by ∞-morphisms

$$\mathfrak{g} \rightsquigarrow \mathfrak{h}$$

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Thm (Konts.) $\exists$ ∞-q-iso of dg-Lie algs

$$\varphi : T_{poly} V \rightsquigarrow C(\mathcal{O}_V, \mathcal{O}_V)[1]$$

The explicit map

$$\overline{MC}(T_{poly} V) \longrightarrow \overline{MC}(C(\mathcal{O}_V, \mathcal{O}_V)[1])$$

$$\alpha \longrightarrow \sum_{k \geq 1} \frac{1}{k!} \varphi^k(\alpha \ldots \alpha)$$

Recall the definition of ∞-morphism of dg-Lie algebras, L_∞-morphisms.

it is a morphism of co-commutative dgcoalgs

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$$S^c(g[1]) \longrightarrow S^c(h[1])$$