A very careful derivation of the P equations

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The h-action equation:

$$\mu // hm^{xy}_z // tha^{uz} = \mu // tha^{ux} // tha^{uy} // hm^{xy}_z$$

The prior spice setup:

$$(\lambda, w) // tha^{ux} := (\lambda, w + P_u(\lambda_x)) // CC_u^{\lambda_x}.$$

Meaning,

$$P_u(\lambda) // CC_u^{\lambda} = J_u(\lambda), \quad \text{or} \quad J_u(\lambda) // C_u^{-\lambda} = P_u(\lambda).$$

$$\frac{d}{d\epsilon}\Big|_0 P_u(\epsilon\lambda) = \frac{d}{d\epsilon}\Big|_0 \left(J_u(\epsilon\lambda) // CC_u^{-\epsilon\lambda}\right) = div_u \lambda$$

The h-action equation becomes:

$$P_u(bch(\lambda_x, \lambda_y)) // CC_u^{bch(\lambda_x, \lambda_y)}$$

$$= P_u(\lambda_x) // CC_u^{\lambda_x} // CC_u^{\lambda_y // CC_u^{\lambda_x}} + P_u(\lambda_y // CC_u^{\lambda_x}) // CC_u^{\lambda_y // CC_u^{\lambda_x}}$$

cancelling the $CC_u^{\lambda_y // CC_u^{\lambda_x}}$ everywhere gives

$$P_u(bch(\lambda_x, \lambda_y)) // CC_u^{\lambda_x} = P_u(\lambda_x) // CC_u^{\lambda_x} + P_u(\lambda_y // CC_u^{\lambda_x})$$

Using $\lambda_x = s\lambda$, $\lambda_y = \epsilon\lambda$, get

Cf. "crossed homomorphisms"

$$P((s+\epsilon)\lambda) // CC_u^{s\lambda} = P(s\lambda) // CC_u^{s\lambda} + P(\epsilon\lambda // CC_u^{s\lambda})$$

Now take an infinitesimal ϵ to get

$$\left(\frac{d}{ds} P(s)\right) // CC_u^{s\lambda} = div_u (\lambda // CC_u^{s\lambda})$$

and $P(0) = 0$, so

$$P_u(t\lambda) = \int_0^t ds \left[\mathrm{div}_u(\lambda // CC_u^{s\lambda}) // u \to C_u^{-s\lambda} \right]$$