Vallette: Operads - definitions, examples, first properties

January-16-13
5:10 AM

See exercise sheet at http://www.newton.ac.uk/programmes/GDO/lectureseries.html#operads
and the Loday-Vallette book at http://math.unice.fr/~brunov/Operades.html.

Recall $\mathrm{End}_V := \{\mathrm{Hom}(V^{\otimes n}, V)\}_{n \in \mathbb{N}}$

1. $\mathrm{End}_V \circ \mathrm{End}_V \longrightarrow \mathrm{End}_V$
2. $$\mathrm{End}_V(k) \otimes \bigotimes_{\alpha=1}^{k} \mathrm{End}_V(i_\alpha) \longrightarrow \mathrm{End}_V(n)$$ $$n = \sum i_\alpha$$
3. $$\mathrm{End}_V(n) \otimes \mathrm{End}_V(m) \xrightarrow{\circ_i} \mathrm{End}_V(n+m-1)$$

Definitions Consider the category of algebras of type P : P-algebras

Ex[1]: P = associative algebras. Free:

$\overline{T}(V) := \bigoplus_{n \geq 1} V^{\otimes n}$ — non-commutative poly w/ no constant term.

Monoid structure: $\overline{T}(\overline{T}V) \longrightarrow \overline{T}(V)$

Note — $V^{\otimes n} = \mathbb{K}[S_n] \otimes_{S_n} V^{\otimes n}$ —

set $Ass(0) = 0$, $Ass(n) = \mathbb{K}(S_n)$

Ex[2]: unital associative —.

Free: $TV := \bigoplus_{n \geq 0} V^{\otimes n}$ $\quad uAss(n) = \_\_\_$

same monad structure.

Ex[3]: commutative Algs.

Ex 4: Unital commutative.

} automatically also associative

Free: polynomials with / without constant term

$\mathrm{Com}_n = \mathbb{k}$ w/ trivial S^n-action,
get $S^n(V) = \mathrm{Com}_n \otimes_{S_n} V^{\otimes n}$

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, bend right, "P:\ \text{Free functor}"'] & P\text{-algs} \arrow[l, "U:\ \text{Forgetful functor}"'] \end{tikzcd}$$

Universal property of Free:

$$\begin{tikzcd} V \arrow[r, "i_V"] \arrow[dr, "\varphi"'] & P(V) \arrow[d, dashed, "\exists!\, \tilde{\varphi}"] \\ & A \quad P\text{-alg} \end{tikzcd}$$

<u>Def</u> $\mathcal{L}_P := UP$ is a monad — a monoid in $(\mathrm{End}_{\mathrm{Vect}}, \circ, \mathrm{id})$ meaning there are nat. trans.

$$\mathcal{L}_P \circ \mathcal{L}_P \xrightarrow{\gamma} \mathcal{L}_P \qquad \text{associative}$$

$$\mathrm{id} \xrightarrow{\eta} \mathcal{L}_P \qquad \text{unit.}$$

more details in video

General claim — any pair of adjoint functors gives rise to a monad.

$$P(V) = \bigoplus_{n \in \mathbb{N}} P(n) \otimes_{\mathbb{k}[S_n]} V^{\otimes n}$$

In general,

$$P(n) = P(\mathbb{k}x_1 \oplus \cdots \oplus \mathbb{k}x_n)_{1,1,1,\ldots,1}$$

Def an $\mathbb{S}$-module is a collection (P_n) of right S_n-modules. That's a category!

Every $\mathbb{S}$-module gives an endo-functor of Vect:

$$\underbrace{\mathcal{L}_P : V \longmapsto \bigoplus_{n \geq 0} P_n \otimes_{S_n} V^{\otimes n}}_{\text{"Schur functors"}}$$

Claim The composition of two Schur functors is a Schur functor.

more on video

Example If P, Q are $\mathbb{S}$-modules,

$$(P \tilde{\otimes} Q)(k) := \bigoplus_{n+m=k} P(n) \otimes Q(m) \otimes_{S_n \times S_m} \mathbb{k}(S_k)$$

has $\mathcal{L}_P \otimes \mathcal{L}_Q = \mathcal{L}_{P \tilde{\otimes} Q}$ [Tensor product of Schurs is Schur]

A similar thing works for compositions:

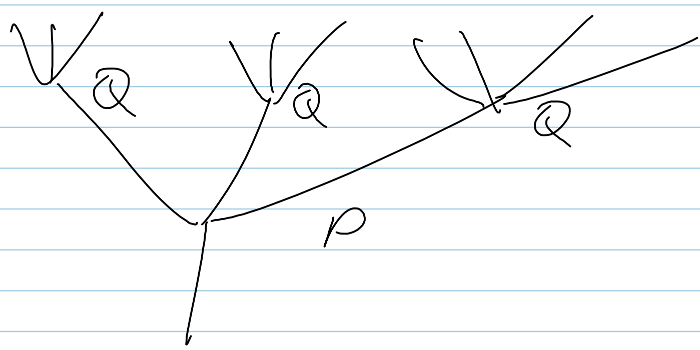
(details in video)

$$(P \tilde{\circ} Q)(k) = \bigoplus_n P(n) \otimes_{S_n} \left[\bigoplus_{i_1 + \cdots + i_n = k} Q(i_1) \otimes \cdots \otimes Q(i_n) \bigotimes_{S_{i_1} \otimes \cdots \otimes S_{i_n}} \mathbb{k} S_k \right]$$

God save abstract algebra -

God save abstract algebra — $\left(\bigotimes_{S_{i_1}\otimes\cdots\otimes S_{i_n}} \mathbb{K}S_k\right)$

what this really means is



S_k acts on the above, mod out by the P symmetry & Q symmetry.

Definition An operad is (P, γ, η) — a monoid in $(\mathbb{S}\text{-mod}, \circ, I)$ "The monoidal definition"

$I = (0, \mathbb{K}, \dots)$ little more in video.

more on signs in video.

There are at least 5 definitions of "an operad":

1. The monoidal def. as above.
2. maps
$$P(n)\otimes\left(\bigotimes_{\alpha=1}^{n} P(i_\alpha)\right) \longrightarrow P\left(\sum i_\alpha\right)$$
(w/ properties)

This is Peter May's classical definition.

3. $\circ_i$ with properties.
4, 5: not today

Def A P-algebra: maps:

$$\begin{array}{ccc} P(A) & \xrightarrow{\gamma_A} & A \\ \| & & \\ \bigoplus_{n\geq 0} P(n)\otimes_{S_n} A^{\otimes n} & \longrightarrow & A \end{array}$$

satisfying .....

Incidently with $A = (A, 0, 0 \ldots)$ regarded as an $\mathbb{S}$-module,

$$P(A) = P \circ A$$

$\mathbb{S}$-module composition as above.

Def A morphism of operads: $F : P \to Q$

$$\begin{array}{ccc} P\circ P & \xrightarrow{F\circ F} & Q\circ Q \\ \downarrow & & \downarrow \\ P & \xrightarrow{F} & Q \end{array}$$

Proposition a P-algebra structure on A is a morphism of operads $P \longrightarrow End_A$.

Can pull-back algebras, given $P \to Q$ from Q to P.

Continued in the afternoon:

The 3 graces:

Lie Ass Com

See video for a discussion of the adjoint functors

$$\begin{tikzcd} \text{Lie-alg} \arrow[r, bend right, "U"'] & \text{Ass-alg} \arrow[l, bend right, "j^*"'] \end{tikzcd}$$

$U: \quad \mathfrak{g} \longmapsto \mathrm{Ass} \circ_{\mathrm{Lie}} \mathfrak{g}$

Co-equalizer:

$$\mathrm{Ass} \circ \mathrm{Lie} \circ \mathfrak{g} \rightrightarrows \mathrm{Ass} \circ \mathfrak{g} \twoheadrightarrow \mathrm{Ass} \circ_{\mathrm{Lie}} \mathfrak{g}$$

Further discussion on video.

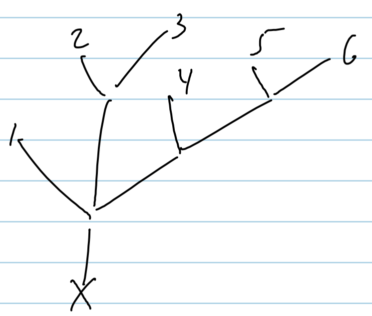
<u>Presentations</u> We need to know "Free operads" & "operadic ideals"

$$\begin{tikzcd} \mathbb{S}\text{-mod} \arrow[r, bend right, "\mathcal{F}ree / \mathcal{T}ree"'] & \text{Operads} \arrow[l, bend right, "U"'] \end{tikzcd}$$

These should be adjoint functors.

Given $E = \{E(n)\}$ in $\mathbb{S}$-mod, let

$\mathcal{T}(E) =$ Trees labeled by E:



unlabeled root
leafs labeled by $1 \ldots n$
internal vertices labeled by E

composition is grafting.

<u>Thm</u> $\mathcal{T}ree$ is $\mathcal{F}ree$.

$\mathcal{I} \subset \mathcal{P}$ is an ideal in an operad if

$$\gamma(i, p_1 \dots p_k) \in \mathcal{I}$$
$$\gamma(p, p_1 \dots i_\ell \dots p_k) \in \mathcal{I}$$

These are exactly the conditions that make $\mathcal{P}/\mathcal{I}$ an operad.

<u>Examples</u> 1. $\mathrm{Ass} = \langle Y \rangle / Y = Y$

So the generators are

$$E = (0, 0, \mathbb{K}(S_2), 0, \dots)$$

2. $\mathcal{L}ie = \langle \overset{1\ 2}{Y} \rangle \Big/ \overset{1\ 2}{Y} + \overset{2\ 1}{Y} = 0 = \overset{1\ 2\ 3}{Y} + \overset{3\ 1\ 2}{Y} + \overset{2\ 3\ 1}{Y}$

or alternatively, use

$$E = (0, 0, \epsilon_2, 0, \dots)$$

That's Bruno's preferred choice.