Vallette: Koszul Duality for Operads, 2

January-30-13
5:28 AM

More on the previous lecture....

$$\mathrm{Hom}_{op}(\mathcal{T}(s^{-1}\bar{\mathcal{C}}),\mathcal{P}) \cong \mathrm{Hom}_{\$}^{-1}(\mathcal{C},\mathcal{P}) \cong \mathrm{Hom}_{coop}(\mathcal{C},\mathcal{T}^c(s\bar{\mathcal{P}}))$$

$$\cup \qquad\qquad\qquad \cup \qquad\qquad\qquad \cup$$

$$\mathrm{Hom}_{dgop}(\Omega\mathcal{C},\mathcal{P}) \cong \mathrm{Tw}(\mathcal{C},\mathcal{P}) \cong \mathrm{Hom}_{dgcoop}(\mathcal{C},B\mathcal{P})$$

$$\cup \qquad\qquad\qquad \cup \qquad\qquad\qquad \cup$$

$$QI(\Omega\mathcal{C},\mathcal{P}) \cong \mathrm{Kos}(\mathcal{C},\mathcal{P}) \cong QI(\mathcal{C},B\mathcal{P})$$

⋮

Things simplify if the operads are "quadratic".

See video, even if this phrase appears here too often.

Yet I'm a low-faith fellow, and there's only that much I can go without knowing that what's being done <u>must</u> be done for the purpose of some good cause. In other words, I cannot work unmotivated.

And, perhaps embarassingly, I still don't understand the motivation for "resolutions", even in the most basic sense.

I guess, even before "what's a resolution", comes "what's a module". I don't even know the answer to that.

<u>Def.</u> Gerstenhaber algebra:

$* : A^{\otimes 2} \to A$ deg 0 commutative product.

$[\,,\,] : (sA)^{\otimes 2} \to sA$ deg 0 bracket.
(skew symmetric)

$$\Updownarrow$$

$\langle\ \ \rangle : A^{\otimes 2} \longrightarrow A$ deg 1 symmetric ....

... The homology of the little disk operad.

Does the little disk operad ever say something truly useful about honest braids?

"The Chevalley-Eilenberg complex of a free Lie algebra is acyclic"

Rewriting methods

Associative algebras

I should try to understand this as "Gaussian elimination in the bulk".

$$A(V,R) = S(V) = T(V)/x\otimes y - y\otimes x$$

1. Choose a totally ordered basis for V

$$V = \{x_1 < x_2 < x_3 < \dots\}$$

2. Consider the graded lexicographic order, or any other order w/ increases under concatenation

$$x_1 < x_2 < x_3$$
$$x_1x_1 < x_1x_2 < \dots$$

Choose a basis for R s.t.

the leading term [highest] in every relation does not appear in any other relation.

3. Interpret the relations as decreasing rewriting rules.

The "critical monomials" are those in which XYZ both XY & YZ are reducible.

4. Write the diamond for each critical monomial

$$
\begin{tikzcd}
 & x_3 x_2 x_1 \arrow[dl, "u\cdot"'] \arrow[dr, "\cdot v"] & \\
x_2 x_3 x_1 \arrow[d] & & x_3 x_1 x_2 \arrow[d] \\
x_2 x_1 x_3 \arrow[d] & & x_1 x_3 x_2 \arrow[d] \\
x_1 x_2 x_3 \arrow[rr, phantom, "="] & & x_1 x_2 x_3
\end{tikzcd}
$$

"Confluence" — going both ways we got to the same place.

Thm If such a procedure exists, which always leads to confluences, then the algebra is Koszul.

A basis A is "words w/ no leading terms"