Vallette: Koszul Duality for Associative Algebras, 2

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<u>Motivation</u> To get a homology theory, "derive a functor".

adga: category of augmented dga's.

$Q : \text{adga} \to \text{Vect}$ by $\bar{A} = \ker \epsilon$

$$(A, \mu, u, \epsilon) \longmapsto A / \mu(\bar{A}, \bar{A})$$

Quillen homology of A:

$$\text{"quasi-free"} \quad \underbrace{(T(X), d)}_{\text{"cofibrant resolution"}} \xrightarrow{\sim} A$$

Example: $D = T(\Delta) / \langle \Delta^2 \rangle \qquad |\Delta| = 1$

$$D = \mathbb{K} \oplus \langle \Delta \rangle \oplus 0 \ldots$$

deg: 0   1   2

Resolve: $T(\underset{\deg 1}{\delta_1}, \underset{\deg 3}{\delta_2}, \underset{\deg 5}{\delta_3}, \ldots) \quad \begin{matrix} \mathbb{K} \longrightarrow \mathbb{K} \\ \delta \longrightarrow \Delta \end{matrix}$

w/ $d\delta_2 = \delta_1^2 \qquad \deg d = -1$!

but δ_1^3 is the image of 2 things, $\pm \delta_1 \delta_2$ & $\pm \delta_2 \delta_1$

So $d\delta_3 = \delta_1 \delta_2 + \delta_2 \delta_1$

$$\cdots \quad d\delta_n = \sum_{k=1}^{n-1} \delta_k \delta_{n-k}$$

So $$H^{Q}_{\cdot}(D) = H_{\cdot}(X, d=0) = X$$

In general,

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[dr, "d_1"] \arrow[dddr, "d_2"'] \arrow[dddddr, "d_3"'] & \bar{T}X \\
& X \\
& \oplus \\
& X^{\otimes 2} \\
& \oplus \\
& X^{3} \\
& \vdots
\end{tikzcd}$$

d defines an ∞-co-algebra structure.

If only d_2 is present, then it is an honest co-algebra structure.

Bar & cobar constructions – recall the "twisting morphisms"

$$\begin{array}{ccccc}
\mathrm{Hom}_{alg}(T(s^{-1}\bar{C}), A) & \cong & \mathrm{Hom}^{\overset{\deg}{-1}}(C;A) & \cong & \mathrm{Hom}_{coalg}(C; T^{c}(s\bar{A})) \\
\cup & & \cup & & \cup \\
\mathrm{Hom}_{dg\,alg}(\Omega C, A) & \cong & Tw(C;A) & \cong & \mathrm{Hom}_{dg\text{-}coalg}(C, BA)
\end{array}$$

So $\quad B:$ aug dga alg $\rightleftarrows$ coaug dga coalg $:\Omega$

"adjunction"

: see video

:

Every $\alpha \in Tw(C;A)$ factors uniquely

$$\overset{i}{\nearrow} \Omega C \overset{f_\alpha}{\searrow}$$

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "g_\alpha"'] & & A \\ & BA \arrow[ur, "\pi"'] & \end{tikzcd}$$

more in video

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3. Kostul morphisms: $\alpha : C \to A$ $|\alpha| = -1$

$$\left(C \otimes A,\ d_C \otimes I + I \otimes d_A + \tilde{d}_\alpha =: d_\alpha\right)$$

$$\tilde{d}_\alpha := C \otimes A \xrightarrow{\Delta} C \otimes C \otimes A \xrightarrow{\alpha} C \otimes A \otimes A \xrightarrow{\mu} C \otimes A$$

Lemma: $(d_\alpha)^2 = \tilde{d}_{\partial\alpha + \alpha \star \alpha}$ ... pictorial proof in video.

So if α is a twisting morphism, then $d_\alpha^2 = 0$.

<u>Def</u> $C \otimes_\alpha A := (C \otimes A, d_\alpha)$ "Twisted tensor product"

<u>Def</u> α is a Kostul morphism if the above is acyclic. $(\sim \mathbb{K})$

The set of such is $\mathrm{Kos}(C,A) \subset \mathrm{Tw}(C,A)$

<u>Thm</u> TFAE (under some grading conditions)

1. $C \otimes_\alpha A$ acyclic
2. $A \otimes_\alpha C$ acyclic
3. $\Omega C \xrightarrow[f_\alpha]{\sim} A$ is quasi-iso
4. $C \xrightarrow[g_\alpha]{\sim} BA$ — " — .

proof in video / w/ some correction.

<u>Corollary</u> $\Omega BA \to A$ is a functorial quasi-free resolution. ....

Koszul duality theory: Suppose A comes w/ a quadratic presentation:

$$(V, R) \quad \text{s.t.} \quad R \subset V^{\otimes 2}$$

$$A = A(V,R) = T(V)/\langle R \rangle$$
$$= \mathbb{k} \oplus V \oplus V^{\otimes 2}/R \oplus \cdots$$

$A(V,R)$ is an initial object in the category of algebras "under TV" which annihilate R.

Dually: $C = C(V,R) = \mathbb{k} \oplus V \oplus R \oplus (R \otimes V \cap V \otimes R)$

$$\cdots \bigcap_{i+j+2=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}$$

a terminal object in the category of C's.

$$\begin{tikzcd} V^{\otimes 2}/R & T^c V \arrow[l, two heads] & C_1 \arrow[l] \arrow[ll, bend right, "0"'] \end{tikzcd}$$

<u>Example 1</u> $V = \mathbb{k}\Delta$, $R = \mathbb{k}\Delta^2$

$$A(V,R) = D = T(\Delta)/\Delta^2 = 0$$

$C(V,R) = {}^{\backprime}|^{\backprime}(V)$

<u>Example 2</u> V, $R = \langle x\otimes y - y\otimes x \rangle$

$A(V,R) = S(V)$

$C(V,R) = \langle \sum_{\sigma\in S_n} (-1)^{\sigma} v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_n} \rangle$

$= \Lambda^c(V)$

<u>Def</u> Koszul dual coalg: $A^{i} = C(sV, s^2R)$

Consider $\kappa : A^{i} \hookrightarrow T^c(sV) \twoheadrightarrow sV \xrightarrow{s^{-1}} V$

$\hookrightarrow T(V) \twoheadrightarrow A.$

<u>Proposition</u> $\kappa \in Tw(A^{i}, A)$

<u>Def</u> (V,R) is Koszul if κ is a Koszul morphism.

<u>Thm</u> TFAE:

1. $A^{i} \otimes_\kappa A$ is acyclic [The Koszul complex]
2. $A \otimes_\kappa A^{i}$ is acyclic
3. $\Omega A^{i} \xrightarrow{\sim} A$ is q.i.
4. $A^{i} \hookrightarrow BA$ is q.i.

Back to example 1 on video...

Back to example 2 —11—

Koszul complex: $\Lambda^c(sV) \otimes S(V)$ — acyclic

Def $A^! = (A^i)^* \ldots$

Proposition When V is f.d.,

$$A^! \cong T(V^*) / \langle R^\perp \rangle$$