Vallette: Koszul Duality for Associative Algebras, 1

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8:51 AM

Bar and cobar constructions:

(A, μ) Associative algebra.

(C, Δ) co-associative co-algebra.

Consider $\mathrm{Hom}(C, A) \ni f, g$

make $f * g \in \mathrm{Hom}(C, A)$ by

$$\begin{tikzcd} & C \arrow[r, "f"] & A \arrow[dr] & \\ C \arrow[ur] \arrow[dr, "\Delta"'] & & & A \\ & C \arrow[r, "g"] & A \arrow[ur] & \end{tikzcd}$$

<u>Claim</u> The convolution product is associative.

Two examples in video.

(A, μ, d_A) a dga (diff. graded alg.)

$|d_A| = -1$

(C, Δ, d_C) a dg-coalg $|d_A| = -1$.

Get $(\mathrm{Hom}(C, A), *, \partial)$, where

$\partial f = d_A \circ f - (-1)^{|f|} f \circ d_C$

MC equation (Maurer-Cartan)

in degree -1: $\quad \partial\alpha + \alpha * \alpha$

The space of sol'ns is $Tw(C, A)$,
"twisting morphisms $C \to A$"
it's a bifunctor.

We want to represent Tw:

$$\mathrm{Hom}(?, A) = Tw(C, A) = \mathrm{Hom}_{dgca}(C, ?)$$

in dga's

First step: $\quad$ homological shift of deg -1 $\quad (s^{-1}C_\bullet)_n = C_{n+1}$

$$\mathrm{Hom}_{alg}(T s^{-1} C, A) \cong \mathrm{Hom}_{-1}(C, A)$$

free alg.

$$\cong \mathrm{Hom}(C, T^c sA)$$

free co-alg

Need a differential on $T(s^{-1}C)$; define

d_2 to be the unique derivation which extends

$$s^{-1}C \xrightarrow{\Delta} s^{-1}C \otimes C \xrightarrow{s^{-1}} s^{-1} \otimes s^{-1} \otimes C \otimes C$$
$$\to (s^{-1}C) \otimes (s^{-1}C) \hookrightarrow T(s^{-1}C)$$

Also, let d_1 be the unique derivation that extends d_C

<u>Prop</u> $(d_1+d_2)^2=0$, making

$(T(s^{-1}C), d_1+d_2)$ a dga.

This is the co-bar construction of C, denoted ΩC

<u>Claim</u>

$$\mathrm{Hom}_{dga}(\Omega C, A) = Tw(C, A)$$

details in video.

On the right hand side: co-everything

$$BA = (T^c(sA), \underset{\substack{\downarrow \\ \text{diff.}}}{d_1} + \underset{\substack{\downarrow \\ \text{use } \mu \text{ on sequences.}}}{d_2})$$

details in video

$$\mathrm{Hom}_{dgca}(C, BA) \cong Tw(C, A)$$

<u>Remark</u>

Assoc alg $\cong$ augmented unital algs

more in video.

$$\mathbb{K} \rightleftarrows_{\epsilon} A$$