Kathryn Hess on "Operads and their (bi)modules in topology"

January-10-13
8:25 AM

I. (Bi)modules over operads

Baby case: $\varphi : R \longrightarrow S$ ring homomorphism makes S an R-bimodule:

$$\begin{tikzcd} R\otimes S \arrow[r, "\varphi\otimes I"] \arrow[rr, bend right, "\lambda_\varphi"'] & S\otimes S \arrow[r, "\mu_S"] & S \end{tikzcd} \qquad \begin{tikzcd} S\otimes R \arrow[r, "I\otimes\varphi"] \arrow[rr, bend right, "\rho_\varphi"'] & S\otimes S \arrow[r, "\mu_S"] & S \end{tikzcd}$$

The associativity of μ_S & φ is homo $\Rightarrow$ these are actions.

The assoc. of μ_S also implies that they are compatible.

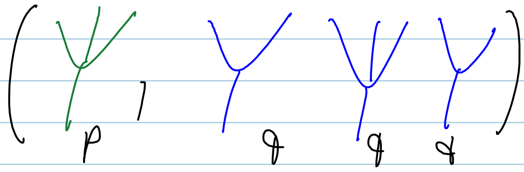
<u>Operad case</u> Recall that an operad is a monoid w.r.t. to Dotsenko's $\circ$.

Let $\varphi : P \longrightarrow Q$ be a morphism of operads.

$$\begin{tikzcd} P\circ Q \arrow[r, "\varphi\circ I"] \arrow[rr, bend right, "\lambda_\varphi"'] & Q\circ Q \arrow[r, "\gamma_Q"] & Q \end{tikzcd}$$

$$\begin{tikzcd} Q\circ P \arrow[r, "I\circ\varphi"] \arrow[rr, bend right, "\rho_\varphi"'] & Q\circ Q \arrow[r, "\gamma_Q"] & Q \end{tikzcd}$$

These have a very different nature!

$\lambda_\varphi :$  $\longrightarrow$

See video

Somehow λ_φ is non-linear in Q while ρ_φ is linear in Q

<u>Defn</u> a left P-module is a (sym) collection $\mathcal{X}$ together w/ a morphism of collections

$$\lambda : P \circ \mathcal{X} \longrightarrow \mathcal{X} \quad \text{s.t. commutes}$$

$$\begin{tikzcd} P \circ P \circ \mathcal{X} \arrow[r] \arrow[d] & P \circ \mathcal{X} \arrow[d] \\ P \circ \mathcal{X} \arrow[r] & \mathcal{X} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x \arrow[r, mapsto] & I \circ x \\ \mathcal{X} \arrow[r] \arrow[dr] & P \circ \mathcal{X} \arrow[d, "\lambda"] \\ & \mathcal{X} \end{tikzcd}$$

A right P-module : $\rho : \mathcal{X} \circ P \longrightarrow \mathcal{X}$ s.t.

$$\begin{tikzcd} \mathcal{X} \circ P \circ P \arrow[r] \arrow[d] & \mathcal{X} \circ P \arrow[d] \\ \mathcal{X} \circ P \arrow[r] & \mathcal{X} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{X} \arrow[r, mapsto] \arrow[dr] & \mathcal{X} \circ P \arrow[d] \\ & \mathcal{X} \end{tikzcd}$$

A bi-module is $\lambda : P \circ \mathcal{X} \longrightarrow \mathcal{X}$

$\& \ \rho : \mathcal{X} \circ Q \longrightarrow \mathcal{X}$

s.t.

$$\begin{tikzcd} P \circ \mathcal{X} \circ Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & P \circ \mathcal{X} \arrow[d] \\ {} & {} \end{tikzcd}$$

$$\mathcal{H} \circ Q \longrightarrow \mathcal{H}$$

<u>Example</u> If A is a P-algebra let

$$Z(A) = (\underset{\text{arity } 0}{A} \; 0 \; 0 \; 0 \quad) \text{ a collection}$$

The $Z(A)$ is a left P-module. details in vid.

<u>Example</u> A an $\mathcal{A}$-algebra (i.e. assoc)

M an (A,A)-bimodule

consider the sequence

$$A \ltimes M = (\underset{\substack{\text{arity}\\ 0}}{A} \; \underset{1}{M} \; 0 \; 0 \; 0 \quad)$$

It's a left $\mathcal{A}$-module! details in vid.

More generally, $\mathcal{H}$ is a left $\mathcal{A}$-module

$\iff$ $\mathcal{H}$ is a graded monoid, that is, an $\mathcal{A}$-algebra w.r.t. the graded tensor product of collections $\odot$:

$$(\mathcal{H} \odot \mathcal{X})(n) = \bigoplus_{k=0}^{n} \mathcal{H}(k) \otimes \mathcal{H}(n-k)$$

Let P be an operad, C be a P-co-algebra:

$$\{ C \otimes P(n) \longrightarrow C^{\otimes n} \} \text{ w/ conditions.}$$

consider $\mathcal{T}(C) = (0 \;\; C \;\; C^{\otimes 2} \;\; C^{\otimes 3} \;\; \ldots)$

Exercise $\mathcal{T}(C)$ is an $(\mathcal{A}, P)$-bimodule.

II Topological examples

Algebra/co-algebra examples: given (X, x_0)

$(\Omega^n X \;\; * \;\; * \;\; - \; - \;)$

is a left $\mathcal{D}_n$-module. $\mathcal{D}_n$: little disk operad.

$(0 \;\; S_* X \;\; (S_* X)^{\otimes 2} \;\; \ldots)$

is an $\mathcal{A}$-$\mathcal{A}$ bimodule.

Operad morphism example: If $m < n$,

$\exists\, \mathcal{D}_m \longrightarrow \mathcal{D}_n \Rightarrow \mathcal{D}_n$ is a

$(\mathcal{D}_m, \mathcal{D}_m)$-bimodule

Goodwillie's Functor calculus

Idea: Given a functor $F: Top_* \longrightarrow Top_*$

which preserves weak equivalences.

* approximate F by "polynomial" functors of higher & higher degree

* decompose poly pieces into homogeneous pieces.
* Try to reconstruct F.

<u>Thm</u> [Goodwillie] Given F.

1. $\forall n \geq 0$ $\exists$ an n-excisive functor $P_nF : Top_* \longrightarrow Top_*$ and nat. trans.

$$\eta_n : F \to P_nF \qquad p_n : P_nF \to P_{n-1}F$$

w/

$$\begin{tikzcd} & P_nF \arrow[d, "p_n"] \\ F \arrow[ur, "\eta_n"] \arrow[r, "\eta_{n-1}"'] & P_{n-1}F \end{tikzcd}$$

and such that $\forall j : F \to G$ w/ G n-excisive, $\exists ! \hat{j}$

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[dr, "\eta_n"'] & G \\ & P_nF \arrow[u, "\hat{j}"'] \end{tikzcd}$$

2. Let $D_nF = hfb(P_nF \xrightarrow{p_n} P_{n-1}F)$ then $\exists$ spectrum $\partial_n F$ w/ a Σ_n-action s.t.

$$D_nF(X) \simeq \Omega^\infty(\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$$

like dividing

The big picture:

by n factorial.

$$\begin{array}{ccc} & & \downarrow \\ D_nF & \longrightarrow & P_nF \\ & & \downarrow \\ & & P_{n-1}F \\ & & \downarrow \\ & & \vdots \\ & & P_1F \\ & & \downarrow \\ & & P_0F \end{array}$$

$\partial_n F =$ nth coefficient

"homogeneous of deg n"

"The Taylor tower of F expanded at $*$"

Remarks 1. We say the tower converges if $FX \simeq \operatorname{holim}_n P_nF(X)$

2. – –

3. The Taylor tower of I is highly non-trivial yet convergent of S.C. spaces.

Thm [Arone-Ching] $\exists$ chain rule:

$\partial_* F = (\partial_n F)_{n \in \mathbb{N}}$ then

$\forall F, G \quad (\partial_* F) \circ (\partial_* G) \longrightarrow \partial_*(F \circ G)$

Corollary Since $I \circ I = I$,

$\partial_* Id$ is an operad in spectra

and $\partial_* F$ is a $\partial_* I$-bimodule

Sheer speculation (Following DBN)

$\mathcal{B}$ = homotopy category of Taylor towers

$\mathcal{B}^{(m)}$ = same, truncated at level m

$\hat{\mathcal{B}} \overset{?}{=}$ homotopy cat. of holims of Taylor towers.

$\mathcal{C}$ = homotopy cat. of $\partial_* I$-bimodules (+ co-alg str. over a certain comonad)

$\mathcal{C}^{(m)}$ = truncated version

$\hat{\mathcal{C}}$ = holim . . . .