John Jones on "Deformation Theory"

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4:34 AM

video on Newton Inst. web site.

Deformation Quantization [From the perspective of symplectic geometry]

Dror's comment: All this should be "unrepresented"

M: Poisson Manifold $C^\infty(M) = A$

$\{\ ,\ \} : A \times A \longrightarrow A$ multilinear

1. $\{f,g\} = -\{g,f\}$
2. Jacobi
3. Leibnitz

$$\{f,gh\} = \{f,g\}h + g\{f,h\}$$

"Poisson Algebra"

In the symplectic situation,

$\omega \in \Gamma(M, \Lambda^2 T^*M)$ makes $P \in \Gamma(M, \Lambda^2 TM)$

$$\{f,g\} = P(df \wedge dg)$$

A $*$-product is a product on $A[[t]]$:

1. Bilinear over $\mathbb{R}[[t]]$.
2. Associative.
3. $a * b = \sum B_n(a,b) t^n$ with
$$B_0(a,b) = a \cdot b \qquad (a, b \in A)$$

A formal deformation quantization of M is a $*$-product on A s.t.

$$\{a,b\} = B_1(a,b) - B_1(b,a)$$

Theorem (Kontsevich) Every Poisson mfd has a (formal) deformation quantization.

Use Gerstenhaber deformation theory:

V: vector space over k.

$C^p(V,V) := \mathrm{Hom}(V^{\otimes p}, V)$

$\circ_i$ composition:

$$(f\circ_i g)(v_1, \dots v_{p+q-1}) = f(v_1, \dots v_{i-1}, g(v_i, \dots v_{i+q-1}), v_{i+q}, \dots v_{p+q-1})$$

$$f\circ g := \sum_i \pm f\circ_i g$$

set $A_f(g,h) = (f\circ g)\circ h - f\circ(g\circ h)$

then $A_f(g,h) = \pm A_f(h,g)$

So $[f,g] = f\circ g - (-1)^{(|f|-1)(|g|-1)} g\circ f$

is a "graded Lie bracket of degree -1".

Example $\mu \in C^2(V,V)$ then μ is associative iff $\mu\circ\mu = 0$ iff (char $k \neq 2$) $[\mu,\mu]=0$

Define $d_\mu : C^p \longrightarrow C^{p+1}$ by

$$d_\mu(x) = \mu\circ x \pm x\circ\mu = [\mu, x]$$

then $d_\mu^2 = 0$ — This is the Hochschild cochain complex of $A=(V,\mu)$.

Apply to construction of a deformation quant:

$$\Theta : A\otimes A \longrightarrow A[[t]]$$

$$\Theta(a,b) = ab + C(a,b)$$

want $(\mu + C)\circ(\mu+C) = 0$

get the Maurer-Cartan eqn's:

$$d_\mu C + C\circ C = d_\mu C + \tfrac{1}{2}[C,C] = 0$$

: We're looking for solutions in

$$C^*(A,A)[[t]]$$

Are there solutions in

$$HH^*(C(A,A)[[t]], d\mu) \quad ?$$

Hochschild–Kostant–Rosenberg (HKR) Theorem:

$$HH^p(A) = \Gamma(M, \Lambda^p TM)$$

$$\{,\} \in HH^2(A,A) \longleftrightarrow p \in \Gamma(M, \Lambda^2 TM)$$

$[\,,\,]_G \longleftrightarrow$ Schouten bracket $[\,,\,]_S$

The Schouten bracket:

1. on vector fields, $[\xi, \eta]_S = [\xi, \eta]$ ← Lie bracket
2. $[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma \pm [\alpha, \gamma] \wedge \beta$

classical computation: $[p,p]_S = 0$

$$d_\mu(p) = 0$$

So we have a solution to MC in homology

<u>Def</u> L_1, L_2 dgL, A quasi-isomorphism $f: L_1 \to L_2$ is a map of dgL's which is an isomorphism in homology.

<u>Claim</u> If L_1 is quasi-iso to L_2, then there is a solution to MC in L_1 iff there is a sol'n in L_2

<u>Kontsevich Formality Thm</u>:

$C^*(A,A)[[t]]$ is quasi-iso to $H^*(A,A)[[t]]$

Deligne's conjecture: see video.
(proven by McClure-Smith, Voronov, Kontsevich-
-Soibelman, ...)