Hidekazu Furusho on p-adic multiple zeta values

January-09-13
7:29 AM

Reference: Furusho p-adic MZV, 2002

$m, k_1 \dots k_m \in \mathbb{N}$

$$\zeta(k_1 \dots k_m) := \sum_{0<n_1<\dots<n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \in \mathbb{R}, \text{ converges. } \quad k_m > 1$$

$\cap$ $\mathbb{Q}_p$ though in $\mathbb{Q}_p$ it diverges. $\leftarrow$ even if $(n_i, p) = 1$

"MPL" = multiple polylog"

$$Li_{k_1 \dots k_n}(z) := \sum_{0<n_1<\dots<n_m} \frac{z^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}}$$

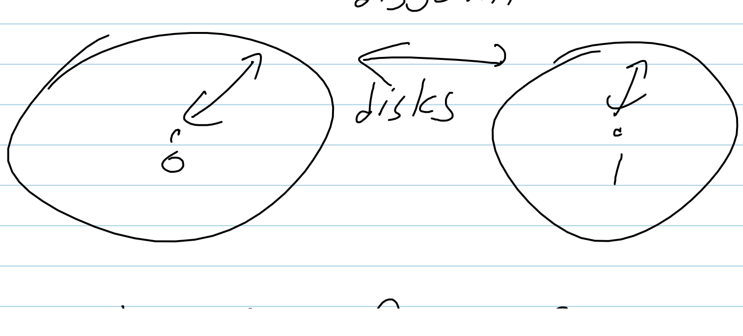
$z \in \mathbb{C}$ converges in unit disk.

ζ is Li @ $z = 1$.

... replace $\mathbb{C}$ by $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$ ($\wedge$ ← top. completion; bar ← alg. completion)

now Li converges of $|z_p| < 1$

In p-adics



So 1 is far from 0. . . . .
. . . need "analytic continuation".

In $\mathbb{C}$:

$$\left\{\begin{array}{l} \dfrac{d}{dz} Li_{k_1\ldots k_m}(z) = \begin{cases} \frac{1}{z} Li_{k_1\ldots k_m - 1} & k_m \neq 1 \\ \frac{1}{1-z} Li_{k_1 \ldots k_{m-1}} & k_m = 1 \end{cases} \\ \\ \dfrac{d}{dz} Li_1(z) = \dfrac{1}{1-z} \end{array}\right.$$

holds also in $\mathbb{C}_p$!

In $\mathbb{C}$

$$Li_1(z) = \int_0^z \frac{dt}{1-t} = -\log(1-z)$$

$$Li_2(z) = \int_0^z Li_1(t)\frac{dt}{t}$$

these allow for analytic cont. to

$$\underbrace{\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}}\qquad \aleph_0 \text{ branches}$$

In $\mathbb{C}_p$, employ Coleman's p-adic integration ('82) attached to a branch, a of p-adic log:

$$\begin{array}{ccc} \log^a: \mathbb{C}_p^\times & \longrightarrow & \mathbb{C}_p^+ \\ \Psi & & \Psi \\ p & \longrightarrow & a \end{array}$$

"

$$Li_1^a(z) = -\log^a(1-z)$$

$$Li_2^a(z) = \int_0^z Li_1^a(t)\frac{dt}{t} \quad \cdots\cdots$$

Get an analytic cont. $pMPL^a$ to $P^1(\mathbb{C}_p) \setminus \{1,\infty\}$ with λ' branches.

In $\mathbb{C}$,

$$\lim_{z \to 1} Li_{k_1 \ldots k_m}(z) = \begin{cases} \text{div} & k_m = 1 \\ \zeta(k_1 \ldots k_m) & k_m \neq 1 \end{cases}$$

In $\mathbb{C}_p$ the same limit converges when $k_m \neq 1$ for any a.

[If $k_m = 1$, sometimes conv. sometimes div.]

Thus we can define

$$\zeta_p^a(k_1 \ldots k_m) \in \mathbb{C}_p$$

<u>Thm</u> These are actually indep of a!

In $\mathbb{C}$, $MZV \in \mathbb{R} \subset \mathbb{C}$

In $\mathbb{C}_p$, $pMZV \in \mathbb{Q}_p$

Examples $\quad \zeta(3) = \zeta(1,2) \quad \zeta_p(3) = \zeta_p(1,2)$

$\zeta(2,1) = \infty \qquad \zeta_p(2,1) = -2\zeta_p(3)$

$\zeta(2k) \in \mathbb{Q}\pi^{2k}$ $\qquad$ $\zeta_p(2k) = 0$

In $\mathbb{C}$, set $Z_0 = \mathbb{Q}$,

$$Z_w = \mathbb{Q}\langle \zeta(k_1 \ldots k_m) : \textstyle\sum k_i = w \rangle \subset \mathbb{R}$$

<u>Dim conj</u> $\dim_{\mathbb{Q}} Z_w = d_w$,

$$d_w = d_{w-2} + d_{w-3} \qquad d_{0,1,2} = 1, 0, 1$$

<u>Thm</u> Deligne-Goncharov, Terasoma

$$\dim Z_w \leq d_w$$

In $\mathbb{C}_p$

<u>Conj</u> $\dim_{\mathbb{Q}} Z_w^p = d_{w-3}$

<u>Thm</u> (Yamashita)

$$\dim_{\mathbb{Q}} Z_w^p \leq d_{w-3}$$

In $\mathbb{C}$

KZ eq'n $\rightsquigarrow$ Drinfel'd associator

$$\Phi_{KZ} \in \mathbb{C}\langle\langle A, B \rangle\rangle$$

$$\| $$

$$1 + \sum (-1)^n \zeta(k_1 \ldots k_m) A^{k_m - 1} B \qquad A^{k_1 - 1} B$$

$$+ \text{ Le-Murakami terms}$$

Do the exact same thing to define Φ_{KZ}^{p}

In $\mathbb{C}$, $\varphi_{KZ}(A,B) := \Phi_{KZ}\left(\frac{A}{2\pi i}, \frac{B}{2\pi i}\right)$

satisfies pent & hex

In $\mathbb{C}_p$, $\Phi_{KZ}^{p} \in GRT_1(\mathbb{Q}_p)$

$$GT_1 \circlearrowright ASSO \circlearrowleft GRT_1$$

philosophical

unclear comments on video

In $\mathbb{C}$

$$ASSO \circlearrowleft GRT_1(\mathbb{C})$$

$$\begin{array}{ccc} \Psi & & \Psi \\ \varphi_{KZ} \ \ \overline{\varphi_{KZ}} & & g \longrightarrow \log g = \sum_{\substack{m \geq 3 \\ m \text{ odd}}} \psi_m \end{array}$$

$g \varphi_{KZ} = \overline{\varphi_{KZ}}$ $\quad$ $\psi_m =$ "Drinfel'd elements"

<u>Remark</u> Brown's result implies that ψ_m generate a free sublie algebra of $\mathfrak{grt}_1(\mathbb{C})$

In $\mathbb{C}_p$,

$$GRT_1(\mathbb{Q}_p) \xrightarrow{\log} \mathfrak{grt}_1(\mathbb{Q}_p)$$

$$\begin{array}{c} \Psi \\ \Phi_{KZ}^{p} \longrightarrow \log \Phi_{KZ}^{p} = \sum_{\substack{n \geq 3 \\ n \in \mathbb{N}}} \psi_n^{p} \end{array}$$