Herbert Gangl on "Multiple Zeta Values"

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10:27 AM

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

* No zeros at $Re(s)=1 \Rightarrow \pi(x) \sim \frac{x}{\log x}$
* $\zeta(1-n) = \frac{-B_n}{n}$; B_n: the nth Bernoulli number, $\sum \frac{B_n t^n}{n!} = \frac{t}{e^t-1}$

$$B_{12} = -\frac{691}{2730}$$

$\zeta(2) = \frac{\pi^2}{6}$ in general $\zeta(2k)$ is easy.

$\zeta(3) \notin \mathbb{Q}$ (Apéry 78)

Little else is known ...

Folklore conjecture: the odd numbers.

$1, \frac{\zeta(3)}{\pi^3}, \frac{\zeta(5)}{\pi^5}, \frac{\zeta(7)}{\pi^7}$ are alg indip over $\mathbb{Q}$.

$$\zeta(a,b) := \sum_{m>n>0} \frac{1}{m^a n^b} \qquad \begin{matrix} b \geq 1 \\ a > 1 \end{matrix}$$

Euler: $\zeta(2,1) = \zeta(3)$ [~32 proofs]

Also

$$\sum_{\substack{\text{convergent}\\ \text{values}}} \zeta(a, w-a) = \zeta(w)$$

$$\zeta(5,2) = 5\,\zeta(2)\zeta(5) + 2\,\zeta(3)\zeta(4) - 11\,\zeta(7)$$

1980's Renaissance: Zagier, Goncharov, Hoffman, Drinfel'd, Kontsevich, Broadhurst.

(1) $\zeta(a,b) + \zeta(b,a) + \zeta(a+b) = \zeta(a)\cdot\zeta(b)$

(2) $\displaystyle\sum_{r=2}^{a+b-1} \left(\binom{r-1}{a-1} + \binom{r-1}{b+1}\right) \zeta(r, a+b-r) = \zeta(a)\zeta(b)$

— all ζ-expressions have the same "weight"

$wt(\zeta(a,b,\ldots)) = a+b+\ldots$

$w(\text{Prod}) = \text{sum of wts.}$

$$\zeta(k_1 \ldots k_d) = \sum_{n_1 > n_2 > \ldots > n_d} \frac{1}{n_1^{k_1} \ldots n_d^{k_d}}$$

d : depth $\quad \sum k_i$: weight.

Relations (1)–(2) generalize.

Rel (1) will use shuffles.

Rel (2) uses K-Z integrals, then shuffles.

Clever idea: introduce also $\zeta(1)$, but if it appears on both sides of an equation, cancel.

Examples:

$$\zeta(1)\zeta(2) = \zeta(1) * \zeta(2) = \zeta(1,2) + \zeta(2,1) + \zeta(2+1)$$

$$\zeta(1)\zeta(2) = \zeta(1) ш \zeta(2) = \zeta(1,2) + 2\zeta(2,1)$$

We now have a candidate for a complete set of relations between MZV's.

$$Z_k = \mathbb{Q}\langle \text{MZV of wt } k \rangle$$

k	1	2	3	4	5	6	7	8
$\dim Z_k$	0	1	2	1	2	2	3	4
		$\zeta(2)$	$\zeta(3)$	$\zeta(4)$	$\zeta(5)$, $\zeta(2,3)$	$\zeta(6)$ $\zeta(3)^2$	$\zeta(7)$ 5·2 3·4	old stuff $+\zeta(3,5)$

old := product of previous things.

Dim Conjecture

$$d_k = \dim Z_k = \text{Coeff. of } t^k \text{ in } \frac{1}{1-t^2-t^3}$$

$$d_k = d_{k-2} + d_{k-3}$$

Thm (Deligne–Goncharov, Terasoma)

$d_k \leq$ the above coeff.

The Hoffman basis conjecture

All MZV are $\mathbb{Q}$-linear comb. of

$\zeta(2 \ldots 2 \ldots 3 \ldots 3 \ldots 2 \ldots 2 \ldots 3)$ only 2's & 3's.

GT connection:

1. Can find the MZV as coeff's of an associator

Conj All MZV rels follow from the associator equations.

Furusho: The associator relations imply the double shuffle relations.

Goncharov–Manin: MZV occur as periods for $\overline{M}_{0,n}$

Brown. The converse is also true.