Frederic Chapoton on "Tree-indexed series, theory and examples"

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1. A natural construction Operad $\xrightarrow{\text{functor}}$ Group.
   Like $(\mathbb{Q}[[x]]^{\text{invertible}}, \circ)$
2. Apply to the PreLie operad, get tree-indexed series.

Operads are in the category Vect.

Used in numerical analysis in the study of Runge-Kutta methods (1960's)
"Butcher Series" they are called.

http://en.wikipedia.org/wiki/Butcher_group

P symmetric operad in $(\mathrm{Vect}_{\mathbb{Q}}, \otimes)$

$P(n) \circlearrowleft S_n$

Assume $P(0) = 0$
$P(1) = \mathbb{Q}\,I$

Composition γ

The free P-algebra on W:

$$P(W) = \bigoplus_{n \geq 1} P(n) \otimes_{S_n} W^{\otimes n}$$

People make an effort to speak in code about simple things.

Has a universal property....

Example: $\mathrm{Com} \longrightarrow \mathrm{Com}(W) = \bigoplus \mathrm{Sym}^m W$

$\mathrm{Ass} \longrightarrow \bigoplus_{n \geq 1} W^{\otimes n}$

w/ $W = \mathbb{Q}$, get

G-invariants

$$P(\mathbb{Q}) = \bigoplus_{n \geq 1} P(n)_{S_n}$$

$$Com(\mathbb{Q}) \cong ASS(\mathbb{Q}) \cong x\mathbb{Q}[x]$$

$\forall a \in P(\mathbb{Q}) \quad \exists\, \varphi_a : P(\mathbb{Q}) \longrightarrow P(\mathbb{Q})$

"substitution operator"

$$\varphi_q \circ \varphi_{q'} = \varphi_{q'(q)} \quad \text{in Com}$$

In general, given $a, b \in P(\mathbb{Q})$, $\exists!\, c$ s.t. $\varphi_a \circ \varphi_b = \varphi_c$. Write $b \circ a := c$

$\circ$ is associative.
$[I]$ is a unit.
have monoid.

$\circ$ is linear only on the left.
$\circ$ is left distributive w.r.t. the P algebra ops.

The computation of $\circ$ is simple.

That's a functor operads $\longrightarrow$ monoids.

To get inverses, switch to the completion

$$\hat{P}(\mathbb{Q}) = \prod_{n \geq 1} P(n)_{S_n}$$

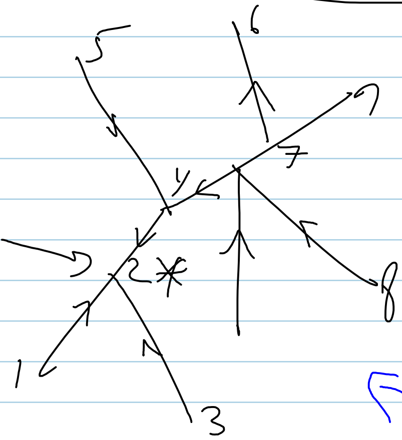
An element of $\hat{P}(\mathbb{Q})$ is invertible iff its degree 1 part is.

Comes from a ~99 paper of Kapranov-Manin.

$\mathrm{PreLie}(n) = \left\langle \begin{array}{c}\text{rooted trees} \\ \text{on } \{1, \dots, n\}\end{array} \right\rangle$

not binary!

root



$|\mathrm{PreLie}(n)| = n^{n-1}$

A pre-Lie algebra is a vector space V with op $\triangleleft : V \otimes V \to V$ s.t.

$$(x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z) = (x \triangleleft z) \triangleleft y - x \triangleleft (z \triangleleft y)$$

Examples 1. Assoc. algebras.
2. Something ... Hochschild.
3. Somehow vector fields on $\mathbb{R}^d$ (later)

$x \triangleleft y = L_x y$

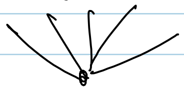
Composition like in acrobat towers

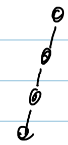
$\widehat{\mathrm{PreLie} \otimes \mathbb{Q}}^{\,inv.}$ (the group of before)

— "The group of tree-indexed series."

computation example on video.

$$\begin{tikzcd} & PreLie(\mathbb{Q}) \arrow[dl, "\text{project on}"'] \arrow[dr, "\text{project on linear trees:}"] & \\ (\mathbb{Q}[[x]], \times) & & (\mathbb{Q}[[x]], \circ) \end{tikzcd}$$





"times"

$\mathbb{R}^d$ $x_1 \dots x_d$ coords

vector fields: $V = \sum_{i=1}^{d} v_i \frac{\partial}{\partial x_i}$

diff. op of order ≤ 1

$$V \circ W = \underbrace{\sum_{i,j} v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j}}_{V \rhd W = W \lhd V} + \sum_{i,j} v_i w_j \frac{\partial^2}{\partial x_i \partial x_j}$$

<u>Claim</u> This is a PreLie product.

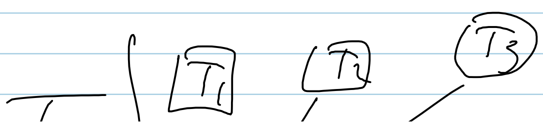
Given any $V \in Vect$ we have

$$PreLie(\mathbb{Q}) \longrightarrow Vect(\mathbb{R}^d)$$

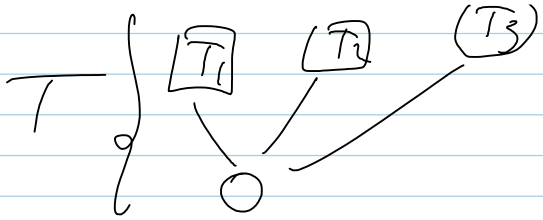
Wow!! so much like KBH!

Fun in video!

Tree-factorial:



$$T! := \underset{\substack{\uparrow \\ \#\text{ verts of } T}}{|T|} \cdot \prod_k T_i!$$



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