Francis Brown on "Periods"

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3:45 AM

Kontsevich & Zagier, "Periods" in "Math unlimited, 2001 & beyond" — strongly recommended!

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \overline{\mathbb{Q}} \subset \mathbb{P} \subset \mathbb{C}$$

"The ring of periods"

Informal definition A period is a complex number whose real & imaginary parts are integrals of rational functions on domains defined by polynomial inequalities.

$$\sqrt{2} = \int_{2x^2 \leq 1} dx \qquad \pi = \iint_{x^2+y^2 \leq 1} dx\,dy \qquad \log 2 = \int_1^2 \frac{dx}{x}$$

$$\zeta(3) = \int_{[0,1]^3} \frac{dx\,dy\,dz}{1-xyz} \qquad \left(\text{all MZV are periods}\right)$$

Conjecture: e is not a period.

Claim Feynman integrals are periods.

Philosophy

$$\begin{array}{c} \overline{\mathbb{Q}} \\ | \\ \mathbb{Q} \end{array} \qquad \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \text{ a profinite group.}$$

There should exist an analogous Galois theory for Periods

$$\begin{array}{c} \mathbb{P} \\ \big|\, G \\ \mathbb{Q} \end{array}$$

G – pro-algebraic group – a limit of matrix groups
"motivic Galois group"

Cohomology: X – smooth scheme of finite type $/\mathbb{Q}$

Betti $\mathbb{Q} \hookrightarrow \mathbb{C}$ $X(\mathbb{C})$ is a smooth mfd, has singular cohomology:

$$H^i_B(X(\mathbb{C}); \mathbb{Q}) \qquad H_i(X(\mathbb{C}); \mathbb{Q})$$

Alg. De-Rham: $H^i_{dR}(X, \mathbb{Q})$

when X is affine,

global regular forms on X

$$H^i_{dR}(X; \mathbb{Q}) \simeq H^i(\Omega^\bullet(X; \mathbb{Q}))$$

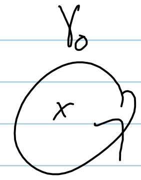
<u>Example</u> $\mathbb{P}^1 \setminus \{0, \infty\} := \mathbb{G}_m$

$$= \operatorname{spec} \mathbb{Q}[z, \tfrac{1}{z}] \qquad \Omega^0(\mathbb{G}_m) = \mathbb{Q}[z, \tfrac{1}{z}]$$

$$0 \longrightarrow \mathbb{Q}[z, \tfrac{1}{z}] \xrightarrow{\quad d \quad} \mathbb{Q}[z, \tfrac{1}{z}]\,dz \longrightarrow 0$$

$$H^0_{dR} = \mathbb{Q} \qquad H^1_{dR} \cong \mathbb{Q}\left[\frac{dz}{z}\right]$$

Betti: $\mathbb{P}_m(\mathbb{C}) = \mathbb{C}^\times \sim S^1 \quad H_1 = \mathbb{Q}[\gamma_0]$



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de-Rham thm

$$\mathrm{Comp}_{dR,B} : H^i_B \otimes \mathbb{C} \xrightarrow{\sim} H^i_{dR} \otimes \mathbb{C}$$

given by the integration pairing.

$$\int_{\gamma_0} \frac{dz}{z} = 2\pi i$$

Relative Cohomology: $Z \overset{i}{\subseteq} X$ subspace

$$0 \to C_\bullet(Z) \to C_\bullet(X) \to C_\bullet(X)/C(Z) \to 0$$

(sing. chains w/ coeffs in $\mathbb{Q}$)

$$H_n(X, Z; \mathbb{Q}) = H\left(C_\bullet(X)/C_\bullet(Z)\right)$$

Long exact seq:

$$H_n(Z) \to H_n(X) \to H_n(X, Z) \to H_{n-1}(Z) \to \ldots$$

Similarly on dR side:

$Z \subset X$ smooth, affine, $\Omega^\bullet_X \xrightarrow{i^*} \Omega^\bullet_Z$

$$H^n_{dR}(X,Z) = H^n(\mathrm{Tot}(\Omega^\bullet_X \to \Omega^\bullet_Z))$$

$$(w_X, w_Z) \in \Omega^n_X \oplus \Omega^{n-1}_Z$$

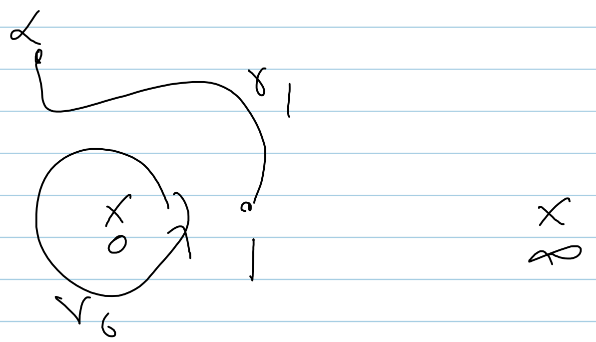
$$\mapsto (dw_X,\ i^* w_X - dw_Z)$$

There is a similar de-Rham comparison more in video

Example "Kummer motive" $\alpha \in \mathbb{Q}^*$

$$\underbrace{\{1,\alpha\}}_{Z} \subset \underbrace{\mathbb{P}^1 \setminus \{0,\infty\}}_{X}$$

more in video



dR: $\frac{dz}{z}$, dz

B: γ_0, γ_1

The period matrix

$$\begin{pmatrix} \int_{\gamma_0} \frac{dz}{z} & \int_{\gamma_1} \frac{dz}{z} \\ \int_{\gamma_0} \frac{dz}{\alpha-1} & \int_{\gamma_1} \frac{dz}{\alpha-1} \end{pmatrix} = \begin{pmatrix} 2\pi i & \log\alpha \\ 0 & 1 \end{pmatrix}$$

log is multivalued, the above matrix is well-def only up to right multiplication by $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$

more in video

Ex $\alpha = 2$, $\log 2 \in \mathbb{P}$ gets replaced by

$$\left(\left[\frac{dz}{z} \in M^{dR}_{\alpha}\right], [\gamma_1] \in M^{B}_{\alpha}\right)$$

Motivic Periods T: Tannkian category
(abelian, $\otimes$, ...) w/ two fiber functors

$$W_B : T \longrightarrow Vec_{\mathbb{Q}} \qquad W_{dR} : T \longrightarrow Vec_{\mathbb{Q}}$$

$$P = \mathrm{Isom}(W_{dR}, W_B)$$

a motivic period is a function $P \to \mathbb{Q}$

more in video

Unipotent GT

$$Z = \mathbb{Q}\langle \zeta(n_1, \dots, n_m) \rangle \quad \text{alg. of MZV}$$

G = Motivic Galois group, group
preserving all (motivic) alg. rels
between MZV's

Known $G \subset GT_1$

Conjecture $G \cong GT_1$

more in video, with much relation to my talk.