Bruno Vallette on "Operads"

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6:31 AM

Rep Theory:

$\underbrace{A}_{\text{Assoc. Alg.}}$ $\quad$ $\underbrace{V}_{\text{V.S.}}$

$$(A, \mu) \longrightarrow (\mathrm{Hom}(V,V), \circ)$$

Ex 1 $\quad D = T(\Delta)/\Delta^2 = 0$

$$\text{rep of } D \iff \begin{cases} \rho(\Delta): V \to V \\ \rho(\Delta)^2 = 0 \end{cases}$$

Ex 2 $\quad X$ top. space

$$V = H^*_{sing}(X, \mathbb{Z}/2\mathbb{Z})$$

$$A_2 = T(Sq^1, Sq^2, \ldots) / \text{Adem relations} \to \mathrm{Hom}(V,V)$$

Multilinear rep. Theory

$$?? \longrightarrow \{\mathrm{Hom}(V^{\otimes n}, V)\}_{n \in \mathbb{N}} =: \mathrm{End}_V$$

1. Collection of v.s. labeled by n
2. Composition maps

$$\mathrm{Hom}(V^{\otimes k}, V) \otimes \mathrm{Hom}(V^{\otimes i_1}, V) \otimes \ldots \otimes \mathrm{Hom}(V^{\otimes i_k}, V)$$

$$\longrightarrow \mathrm{Hom}(V^{\otimes \sum i_j}, V)$$

3. Obvious associativities, identities.

This is the definition of an (non-symmetric) operad!

Example 0: An ordinary algebra becomes

$P(1) = A$

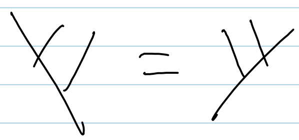
$P(n) = \{0\} \quad n \neq 1$

<u>Example 1</u> $As(n) = \begin{cases} \mathbb{K}\mu_n \ (1\text{-dim}) & n > 0 \\ 0 & n = 0 \end{cases}$

<u>Def</u> A P-algebra structure on V is a morphism of operads $P \to End_V$

An AS-algebra is a (unital) associative algebra.

<u>Claim</u> As is the free operad on  modulo



Question What do you call the "meta" version of a P-algebra?

<u>Def</u> Symmetric operads.

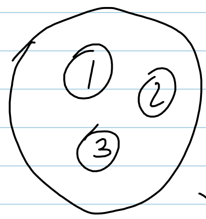
<u>Example</u>

$Com(n) = \begin{cases} \mathbb{K}\mu_n & n \geq 1 \end{cases}$ (trivial S_n action)

$$Com(n) = \begin{cases} \mathbb{K}\circlearrowleft_n \leftarrow \text{action} & n \geq 1 \\ 0 & n = 0 \end{cases}$$

a com-algebra is a commutative algebra.

Can do operads with $(Vect, \otimes)$ replaced by any symm. monoidal category.

Example $\mathcal{D}(n) = \{$  $\}$

"the Mother of all top. operads"

The "little disk operad" is a topological operad.

$\Omega^2 X = Hom((D^2, S^1) \to (X, *))$ is a $\mathcal{D}$-algebra.

Thm [Boardman-Vogt, May] Any $\mathcal{D}$-algebra is homotopy equiv. to a double loop space.

Thm [F. Cohen, 76']

$H_*(\mathcal{D})$ is the Gerstenhaber operad.

Deligne's Conjecture (vague statement)

$C^\bullet(A, A)$ has an action of an operad equivalent to $C^\bullet(\mathcal{D})$

Thm [Fresse 12'] Some operadic description of GT.

More in video.

stuff on Koszul duality in video.