

Solving the 3x5 torus puzzle

Pensieve Header: Solving the 3x5 torus puzzle, with care for lengths and meanings.

```

n = 15; $RecursionLimit = 2^16;
Feed[M[Range[n], ___]] := {};
CycPerm[c___] := Range[n] /. Thread[{c} → RotateRight[{c}]];

CycPerm[1, 2, 3]
{3, 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

gens = {
  M[CycPerm[1, 2, 3, 4, 5], {r1}, 1],
  M[CycPerm[6, 7, 8, 9, 10], {r2}, 1],
  M[CycPerm[11, 12, 13, 14, 15], {r3}, 1],
  M[CycPerm[1, 6, 11], {c1}, 1],
  M[CycPerm[2, 7, 12], {c2}, 1],
  M[CycPerm[3, 8, 13], {c3}, 1],
  M[CycPerm[4, 9, 14], {c4}, 1],
  M[CycPerm[5, 10, 15], {c5}, 1]
};

gens = {
  M[CycPerm[1, 4, 7, 10, 13], {r1}, 1],
  M[CycPerm[2, 5, 8, 11, 14], {r2}, 1],
  M[CycPerm[3, 6, 9, 12, 15], {r3}, 1],
  M[CycPerm[1, 2, 3], {c1}, 1],
  M[CycPerm[4, 5, 6], {c2}, 1],
  M[CycPerm[7, 8, 9], {c3}, 1],
  M[CycPerm[10, 11, 12], {c4}, 1],
  M[CycPerm[13, 14, 15], {c5}, 1]
}

{M[{13, 2, 3, 1, 5, 6, 4, 8, 9, 7, 11, 12, 10, 14, 15}, {r1}, 1],
 M[{1, 14, 3, 4, 2, 6, 7, 5, 9, 10, 8, 12, 13, 11, 15}, {r2}, 1],
 M[{1, 2, 15, 4, 5, 3, 7, 8, 6, 10, 11, 9, 13, 14, 12}, {r3}, 1],
 M[{3, 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}, {c1}, 1],
 M[{1, 2, 3, 6, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15}, {c2}, 1],
 M[{1, 2, 3, 4, 5, 6, 9, 7, 8, 10, 11, 12, 13, 14, 15}, {c3}, 1],
 M[{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 10, 11, 13, 14, 15}, {c4}, 1],
 M[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 13, 14}, {c5}, 1]}

```

Program 5

```

Clear[s, M, T]; TC = 0;
M /: M[a1_, {w1___}, m1_] ** M[a2_, {w2___}, m2_] := M[a1[[a2]], {w1, w2}, m1 + m2];
M /: Inverse[M[a_, w_, m_]] := M[Ordering[a], -Reverse[w], m];
Feed[M[Range[n], ___]] := {};
Feed[M[a_, {w___}, m_]] := Module[
  {i, j, sij, k, l, skl},
  For[i = 1, a[[i]] == i, ++i; j = a[[i]];
  If[Head[sij = s[i, j]] === Integer,
    (* then *) If[m ≥ T[sij][[3]],
      Feed[Inverse[ReplacePart[T[sij], {sij}, 2]] ** M[a, {w}, m]],
      T[s[i, j] = ++TC] = M[a, {w}, m];
      Feed[Inverse[M[a, {w}, m]] ** ReplacePart[T[sij], {sij}, 2]]
    ],
    (* else *) T[s[i, j] = ++TC] = M[a, {w}, m];
  Do[
    If[Head[skl = s[k, l]] == Integer,
      Feed[ReplacePart[T[sij] ** T[skl], {sij, skl}, 2]];
      Feed[ReplacePart[T[skl] ** T[sij], {skl, sij}, 2]]
    ],
    {k, n}, {l, n}
  ]
];
Images[i_] := Prepend[Select[Range[n], Head[s[i, #]] === Integer &], i];
MoveCount[i_, i_] := 0;
MoveCount[i_, j_] := T[s[i, j]][[3]];
TMC[] := Sum[Total[MoveCount[i, #] & /@ Images[i]], {i, n}];
Optimize[] := Timing[
  Do[
    If[Head[sij = s[i, j]] == Integer, Do[
      If[Head[skl = s[k, l]] == Integer,
        Feed[ReplacePart[T[sij] ** T[skl], {sij, skl}, 2]]
      ], {k, n}, {l, n}]],
    {i, n}, {j, n}];
  TMC[]
];

```

```

g = 0;
Print[Timing[
  (++g; Feed[#]; Feed[Inverse[#]]; Product[Length[Images[i]], {i, n}]) & /@ gens
]];
Print[tmc = TMC[]];
While[
  Last[opt = Optimize[]] ≠ tmc,
  tmc = Last[opt];
  Print[opt]
]
{3.260421,
 {5, 25, 125, 653 837 184 000, 653 837 184 000, 653 837 184 000, 653 837 184 000, 653 837 184 000}}
42 059
{1.840812, 334}

15! / 2
653 837 184 000

```

The Worst Case Scenario

```
Table[MoveCount[i, #] & /@ Images[i], {i, n}] // ColumnForm
```

```

{0, 1, 1, 1, 2, 2, 2, 3, 3, 2, 3, 3, 1, 2, 2}
{0, 3, 2, 1, 2, 3, 2, 3, 3, 2, 3, 2, 1, 2}
{0, 2, 2, 1, 3, 3, 2, 3, 3, 2, 2, 2, 1}
{0, 1, 1, 4, 3, 3, 6, 5, 5, 6, 5, 5}
{0, 3, 3, 4, 3, 5, 6, 5, 5, 6, 5}
{0, 3, 3, 4, 5, 5, 6, 5, 5, 6}
{0, 1, 1, 4, 3, 3, 6, 5, 5}
{0, 3, 3, 4, 3, 5, 6, 5}
{0, 3, 3, 4, 5, 5, 6}
{0, 1, 1, 4, 3, 3}
{0, 3, 3, 4, 3}
{0, 3, 3, 4}
{0, 1, 1}
{0}
{0}

```

```
Sum[Max[MoveCount[i, #] & /@ Images[i]], {i, n}]
```

```
58
```

```
TC
```

```
1717
```