Braids on a torus, a cubic group?

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8:53 AM

From Philippe Humbert's thesis, pages 30-31:

2.1 The Lie algebra $\mathfrak{t}_{1,n}$

We define the graded Lie algebra $\mathfrak{t}_{1,n}$, which has been introduced by Bezrukavnikov [11] as the Lie algebra associated with the lower central series of the pure braid group of the torus.

Definition 2.1.1. Let $\mathfrak{t}_{1,n}$ be the graded Lie algebra presented by the degree one generators v_i (for any $v \in H_1$ and $i \in \{1, \dots, n\}$), the degree two generators t_{ij} (for any $i \neq j \in \{1, \dots, n\}$), the linearity relation $(v + \lambda w)_i = v_i + \lambda w_i$ and the following relations (2.1.1-2.1.3) for any $v, w \in H_1$ and any distinct $i, j, k \in \{1, \dots, n\}$.

$$[v_i, w_j] = \langle v, w \rangle t_{ij}, \tag{2.1.1}$$

$$[v_i, t_{jk}] = 0, \tag{2.1.2}$$

$$[x_i, y_i] = -\sum_{j \neq i} t_{ij}. \tag{2.1.3}$$

Lemma 2.1.1. *The relations of Definition 2.1.1 imply that $\sum_{j=1}^n v_j$ is central in $\mathfrak{t}_{1,n}$, and*

$$t_{ij} = t_{ji}, \quad [v_i + v_j, t_{ij}] = 0,$$

as well as the infinitesimal pure braids relations

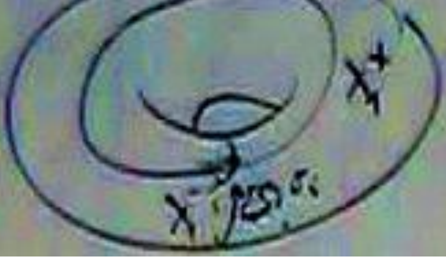
$$[t_{ij}, t_{kl}] = 0 \quad \textit{and} \quad [t_{ij}, t_{ik} + t_{kj}] = 0.$$

In particular, there is a Lie algebra morphism $\mathfrak{t}_n \to \mathfrak{t}_{1,n}$ sending $t_{ij} \in \mathfrak{t}_n$ to $t_{ij} \in \mathfrak{t}_{1,n}$. This morphism multiplies the degree by two.

Proof. Relations (2.1.3) and (2.1.1) imply $[x_i, \sum_{j=1}^n y_j] = 0$, and from (2.1.1), we also have $[y_i, \sum_{j=1}^n y_j] = 0$. Since the x_i's and the y_i's generate $\mathfrak{t}_{1,n}$, it follows that $\sum_{j=1}^n y_j$ is central. Similarly, we show that $\sum_{j=1}^n x_j$ is central. Hence, $\sum_{j=1}^n v_j$ is central for any v. The relation $t_{ij} = t_{ji}$ follows from $t_{ij} = [x_i, y_j] = -[y_j, x_i] = -\langle y, x \rangle t_{ji} = t_{ji}$. Using (2.1.2), we have $[v_i + v_j, t_{ij}] = [\sum_{s=1}^n v_s, t_{ij}] = 0$ and $[t_{ij}, t_{kl}] = [t_{ij}, [x_k, y_l]] = 0$. Last, we have $[t_{ij}, t_{ik} + t_{kj}] = [t_{ij}, [x_i, y_k] + [x_j, y_k]] = [t_{ij}, [x_i + x_j, y_k]] = -[x_i + x_j, [y_k, t_{ij}]] - [y_k, [t_{ij}, x_i + x_j]] = 0.$ □

From Enriquez' June 2010 talk in Montpellier, http://drorbn.net/dbnvp/Enriquez-1006.php:

① $B_{1,n} = \pi_1\left(\frac{T^n\text{-drags}}{S_n}\right) = \langle X_1^{\pm}, \sigma_1, \dots, \sigma_{n-1} \rangle$ /



$\sigma_{n-1}\rangle$ /

$$\left.\begin{array}{l} (\sigma_1^{\pm 1} X_1^{\pm})^2 = (X_1^{\pm} \sigma_1^{\pm 1})^2 \\ (X_1^-, (X_2^+)^{-1}) = \sigma_1^2 \\ (X_1^+, \sigma_i) = 0 \quad i > 1 \\ X_1^{\pm} \cdots X_n^{\pm} = 0 \end{array}\right\} \quad X_i^{\pm} = \sigma_i^{\pm 1} X_{i-1} \sigma_i^{\pm 1}.$$

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