Recover the abstract?

b''h

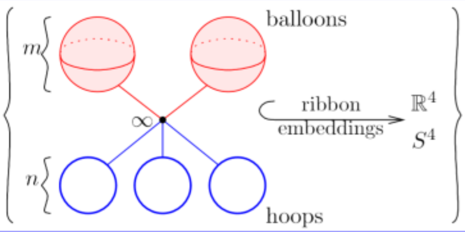
# Balloons and Hoops and their Universal Finite–Type Invariant, BF Theory, and an Ultimate Alexander Invariant

Dror Bar–Natan in Hamburg, August 2012
http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

B

**Scheme.** • Balloons and hoops in $\mathbb{R}^4$, operations and relations with 3D.
• An ansatz for an invariant: computable, related to finite-type and to BF.
• Reduction to an "ultimate Alexander invariant".

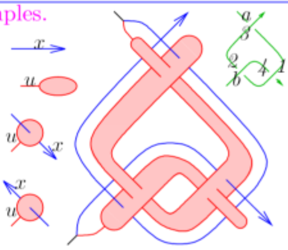
**$\mathcal{K}^{bh}(m,n)$.**




**Examples.**

$\epsilon_x$:
$\epsilon_u$:
$\rho^+_{ux}$:
$\rho^-_{ux}$:

relax the orientations



```
T0 = Rm[3, a] Rp[b, 2] Rp[1, 4];
ζ = T0 // dm[1, 2, 1] // dm[b, 4, b] //
    dm[a, 1, a] // dm[a, 3, a];
ζ[(5)]
```

$\mu\Big[\mathrm{CWS}\Big[0, -[ab], -\frac{[aab]}{2}-\frac{[abb]}{2}, -\frac{[aaab]}{6}+\frac{7[aabb]}{4}-\frac{5[abab]}{2}-\frac{[abbb]}{6}, -\frac{[aaaab]}{24}+\frac{11[aaabb]}{12}-\frac{3[aabab]}{2}+\frac{11[aabbb]}{12}-\frac{3[ababb]}{2}-\frac{[abbbb]}{24}\Big],$
$h[b]\,\mathrm{LS}\Big[\langle a\rangle, -\langle ab\rangle, -\langle aab\rangle+\frac{\langle abb\rangle}{2}, -\frac{\langle aaab\rangle}{2}+\frac{3\langle aabb\rangle}{2}-\frac{\langle abbb\rangle}{6}, -\frac{\langle aaaab\rangle}{6}+\frac{7\langle aaabb\rangle}{4}-\frac{7\langle aabbb\rangle}{6}-\langle ababb\rangle+\frac{\langle abbbb\rangle}{24}\Big]+$
$h[a]\,\mathrm{LS}\Big[-\langle a\rangle+\langle b\rangle, \frac{\langle ab\rangle}{2}, \frac{\langle aab\rangle}{12}-\frac{13\langle abb\rangle}{12}, -\frac{25\langle aabb\rangle}{24}+\frac{\langle abbb\rangle}{2}, -\frac{\langle aaaab\rangle}{720}-\frac{91\langle aaabb\rangle}{180}+\frac{59\langle aabab\rangle}{360}+\frac{143\langle aabbb\rangle}{90}+\frac{77\langle ababb\rangle}{40}-\frac{119\langle abbbb\rangle}{720}\Big]\Big]$

A

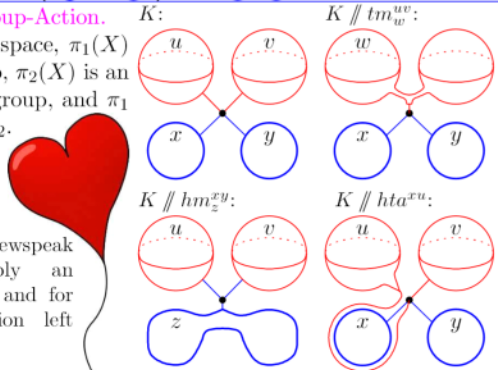
• u-Knots inject into $\mathcal{K}^{bh}$ (likely u-tangles too).
• In fact, $n$-component v/w-tangles map onto $\mathcal{K}^{bh}(n,n)$; the kernel contains Reidemeister moves and the "overcrossings commute" relation, and conjecturally, that's all.


Teichner by Bergman

**Connected Sums.**

makes sense!

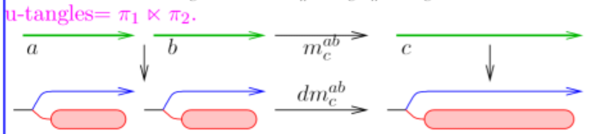
**Meta-Group-Action.** If $X$ is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and $\pi_1$ acts on $\pi_2$.




("$/\!\!/$" is newspeak for "apply an operator" and for "composition left to right")

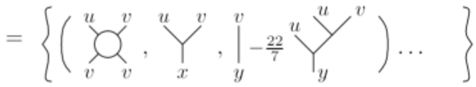
**Properties.**
• Associativities: $m^{xy}_x /\!\!/ m^{xz}_x = m^{yz}_y /\!\!/ m^{xy}_x$, for $m = tm, hm$.
• Action axiom $t$: $tm^{uv}_w /\!\!/ hta^{xw} = hta^{xu} /\!\!/ hta^{xv} /\!\!/ tm^{uv}_w$,
• Action axiom $h$: $hm^{xy}_z /\!\!/ hta^{zu} = hta^{xu} /\!\!/ hta^{yu} /\!\!/ hm^{xy}_z$.
• SD Product: $dm^{ab}_c := hta^{ba} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$ is associative.

**u-tangles**$=\pi_1 \ltimes \pi_2$.




**The Meta-Group-Action $M$.** Let $T$ be a set of "tail labels" ("balloon colours"), $H$ a set of "head labels" ("hoop colours"), and let $\{h_x\}_{x\in H}$ be a set of formal symbols. Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators $T$ and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on $T$, so there's $tr : FA \to CW$, and both $FA$ and $CW$ are $FL$-modules. Let

$$M(T,H) := \{\mu = (\omega, \bar\lambda = (\lambda_x)_{x\in H}) :\ \omega \in CW,\ \lambda_x \in FL\}$$



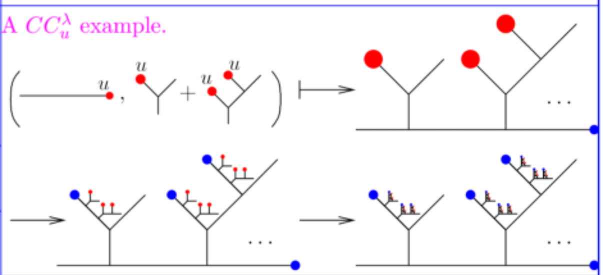

**Operations.** Set $(\omega_1,\bar\lambda_1)(\omega_2,\bar\lambda_2) := (\omega_1+\omega_2, (\bar\lambda_1,\bar\lambda_2))$ and with $\mu = (\omega,\bar\lambda)$ define

$$tm^{uv}_w : \mu \mapsto \mu /\!\!/ (u, v \mapsto w),$$

$$hm^{xy}_z : \mu \mapsto \Big(\omega, \big(\dots, \widehat{\lambda_x}, \widehat{\lambda_y}, \dots, \mathrm{bch}(\lambda_x,\lambda_y)_z\big)\Big)$$

$$hta^{xu} : \mu \mapsto \underbrace{\mu \overset{\text{"stable apply"}}{\overbrace{/\!\!/\!\!/}} (u \mapsto e^{\mathrm{ad}\,\lambda_x}(\bar u)) /\!\!/ (\bar u \mapsto u)}_{\mu /\!\!/ CC^{\lambda_x}_u} + \underbrace{(J_u(\lambda_x), 0)}_{\text{the "}J\text{-spice"}}$$

**A $CC^\lambda_u$ example.**

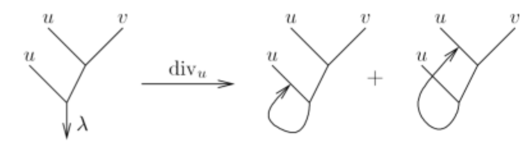



**The Meta-Cocycle $J$.** Set $J_u(\lambda) := J(1)$ where

$$J(0) = 0, \qquad \lambda_s = \lambda /\!\!/ CC^{s\lambda}_u,$$

$$\frac{dJ(s)}{ds} = (J(s) /\!\!/ \mathrm{der}(u \mapsto [\lambda_s, u])) + \mathrm{div}_u \lambda_s,$$

and where $\mathrm{div}_u \lambda := \mathrm{tr}(u\sigma_u(\lambda))$, $\sigma_u(v) := \delta_{uv}$, $\sigma_u([\lambda_1,\lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$ and $\iota$ is the inclusion $FL \hookrightarrow FA$:




**Claim.** $CC^{\mathrm{bch}(\lambda_1,\lambda_2)}_u = CC^{\lambda_1}_u /\!\!/ CC^{\lambda_2 /\!\!/ CC^{\lambda_1}_u}_u$ and

$$J_u(\mathrm{bch}(\lambda_1,\lambda_2)) = J_u(\lambda_1) /\!\!/ CC^{\lambda_2 /\!\!/ CC^{\lambda_1}_u}_u + J_u(\lambda_2 /\!\!/ CC^{\lambda_1}_u),$$

and hence $tm$, $hm$, and $hta$ form a meta-group-action.

**Why ODEs?** **Q.** Find $f$ s.t. $f(x+y) = f(x)f(y)$.
**A.** $\frac{df(s)}{ds} = \frac{d}{d\epsilon}f(s+\epsilon) = \frac{d}{d\epsilon}f(s)f(\epsilon) = f(s)C$.
Now solve this ODE using Picard's theorem or power series.


Alekseev Torossian

A. Add X → X, OC but not UC

B. Add a section about the fundamental group of the complement.

# Balloons and Hoops and their Universal Finite–Type Invariant, 2

**The Invariant $\zeta$.** Set $\zeta(\rho^\pm) = (0, \pm u_x)$. This at least defines an invariant of u/v/w-tangles, and if the topologists will deliver a "Reidemeister" theorem, it is well defined on $\mathcal{K}^{bh}$.

Signs

$\zeta$: $u \;\; x \longmapsto \left(0, +\left|\begin{smallmatrix} u \\ x\end{smallmatrix}\right.\right)$ $\quad u \;\; x \longmapsto \left(0, -\left|\begin{smallmatrix} u \\ x\end{smallmatrix}\right.\right)$

**Theorem.** $\zeta$ is (the log of) a universal finite type invariant (a homomorphic expansion) of w-tangles.

} and BF theory.

Cattaneo

need an aside on how FL/CW parametrize formulas in f.d. Lie algebras.

Also a word about the Alexander relation

**The $\beta$ quotient.** Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central $R$ and with $[u,v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to (\omega, \bar\lambda) \quad \text{with } \omega \in R, \quad \bar\lambda = \sum_{x\in H,\, u\in T} \lambda_{ux} ux,$$

$$\mathrm{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v}-1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u /\!/ CC_u^\lambda = \left(1 + c_u\lambda_u\frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1}\left(e^{c_\lambda}u - c_u\frac{e^{c_\lambda}-1}{c_\lambda}\sum_{v\neq u}\lambda_v v\right),$$

$\mathrm{div}_u\,\lambda = c_u\lambda_u$, and the ODE for $J$ integrates to

$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda}c_u\lambda_u\right),$$

so $\zeta$ is formula-computable to all orders! Can we simplify?

Repackaging.

**$\beta$ Calculus.** Let $\beta(H,T)$ be

$$\left\{\begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l}\omega \text{ and the } \alpha_{ux}\text{'s are ratio-}\\ \text{nal functions in variables}\\ t_u, \text{ one for each } u \in T.\end{array}\right\},$$

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha + \beta + \langle\alpha\rangle\beta & \gamma \end{array},$$

$$hta^{xu} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

Polynomiality, efficiency, categorification

Gukov Morrison

Further directions.

v-knots? AT & KV?

Jones? E-K?

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas