Hamburg bh Handout as of August 8 (evening)

August-08-12
8:26 PM

Not enough green!

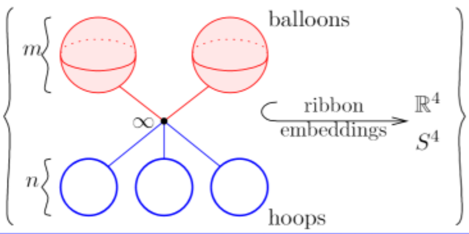
# Balloons and Hoops and their Universal Finite–Type Invariant, BF Theory, and an Ultimate Alexander Invariant

b"h

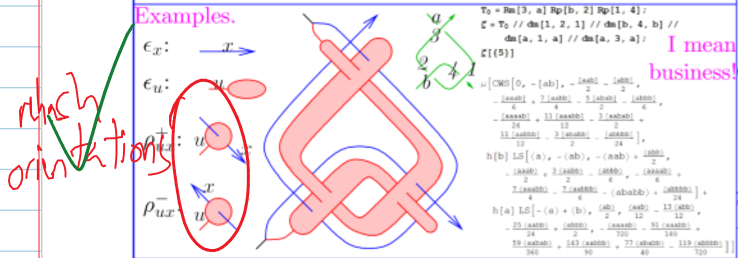
Dror Bar–Natan in Hamburg, August 2012
http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

**Scheme.** • Balloons and hoops in $\mathbb{R}^4$, operations and relations with 3D.
• An ansatz for an invariant: computable, related to finite-type and to BF.
• Reduction to an "ultimate Alexander invariant".

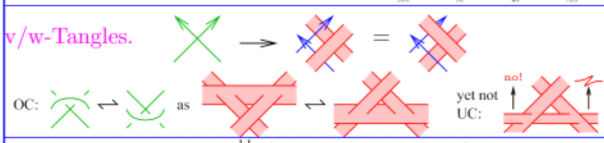
**$\mathcal{K}^{bh}(m,n)$.**




**Examples.**




**v/w-Tangles.**

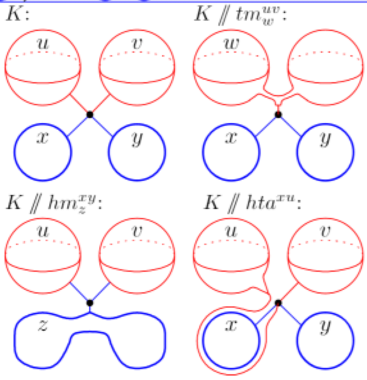



• u-Knots inject into $\mathcal{K}^{bh}$ (likely u-tangles too).
• In fact, $n$-component v/w-tangles map onto $\mathcal{K}^{bh}(n,n)$; the kernel contains Reidemeister moves and the "overcrossings commute" relation, and conjecturally, that's all.


Teichner by Bergman

**Connected Sums.**

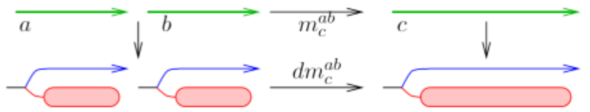

**Meta-Group-Action.** If $X$ is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and $\pi_1$ acts on $\pi_2$.



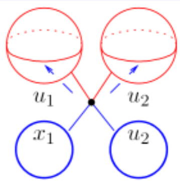

("$/\!\!/$" is newspeak for "apply an operator" and for "composition left to right")

**Properties.**
• Associativities: $m^{xy}_x /\!\!/ m^{xz}_x = m^{yz}_y /\!\!/ m^{xy}_x$, for $m = tm, hm$.
• Action axiom $t$: $tm^{uv}_w /\!\!/ hta^{xw} = hta^{xu} /\!\!/ hta^{xv} /\!\!/ tm^{uv}_w$.
• Action axiom $h$: $hm^{xy}_z /\!\!/ hta^{zu} = hta^{xu} /\!\!/ hta^{yu} /\!\!/ hm^{xy}_z$.
• SD Product: $dm^{ab}_c := hta^{ba} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$ is associative.

**Tangle compositions** $\to \pi_1 \ltimes \pi_2$.




**Invariant #0.** With $\Pi_1$ denoting "honest $\pi_1$", map $\gamma \in \mathcal{K}^{bh}(m,n)$ to the triple $(\Pi_1(\gamma^c), (u_i), (x_j))$, where the meridian of the balls $u_i$ normally generate $\Pi_1$, and the "longtitudes" $x_j$ are some elements of $\Pi_1$. $*$ acts like $*$, $tm$ acts by "merging" two meridians/generators, $hm$ acts by multiplying two longtitudes, and $hta^{xu}$ acts by "conjugating a meridian by a longtitude":



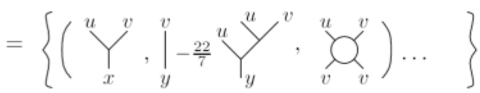

Not computable! (but nearly)

$$(\Pi, (u, \ldots), (x, \ldots)) \mapsto (\Pi * \langle \bar{u} \rangle / (u = x\bar{u}x^{-1}), (\bar{u}, \ldots), (x, \ldots))$$

Can we write the $x$'s as free words in the $u$'s?

**The Meta-Group-Action $M$.** Let $T$ be a set of "tail labels" ("balloon colours"), $H$ a set of "head labels" ("hoop colours"), and let $\{h_x\}_{x \in H}$ be a set of formal symbols. Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators $T$ and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on $T$, so there's $\mathrm{tr} : FA \to CW$, and both $FA$ and $CW$ are $FL$-modules. Let

$$M(T,H) := \{(\bar{\lambda} = (\lambda_x)_{x \in H}, \omega) : \lambda_x \in FL, \omega \in CW\}$$



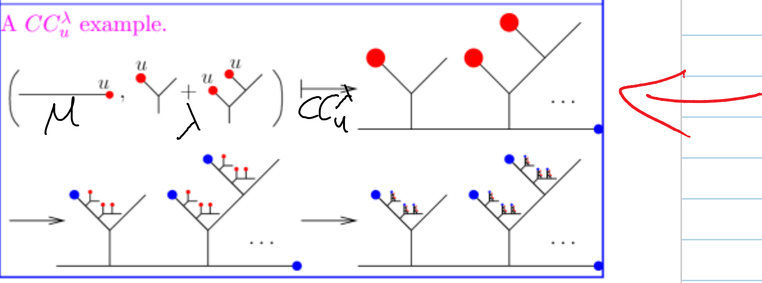

**Operations.** Set $(\bar{\lambda}_1, \omega_1) * (\bar{\lambda}_2, \omega_2) := (\bar{\lambda}_1, \bar{\lambda}_2, \omega_1 + \omega_2)$ and with $\mu = (\bar{\lambda}, \omega)$ define

$$tm^{uv}_w : \mu \mapsto \mu /\!\!/ (u, v \mapsto w),$$

$$hm^{xy}_z : \mu \mapsto \left( \left( \ldots, \widehat{\lambda_x}, \widehat{\lambda_y}, \ldots, \mathrm{bch}(\lambda_x, \lambda_y)_z \right), \omega \right)$$

$$hta^{xu} : \mu \mapsto \underbrace{\mu \overbrace{/\!\!/\!\!/}^{\text{"stable apply"}} (u \mapsto e^{\mathrm{ad}\,\lambda_x}(\bar{u})) /\!\!/ (\bar{u} \mapsto u)}_{\mu /\!\!/ CC^{\lambda_x}_u} + \underbrace{(0, J_u(\lambda_x))}_{\text{the "}J\text{-spice"}}$$

**A $CC^{\lambda}_u$ example.**



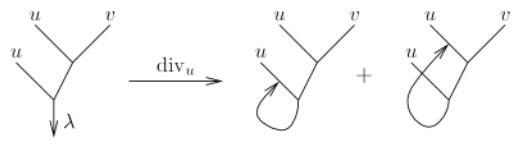
# Balloons and Hoops and their Universal Finite–Type Invariant, 2

**The Meta-Cocycle $J$.** Set $J_u(\lambda) := J(1)$ where

$$J(0)=0, \qquad \lambda_s = \lambda \mathbin{/\!\!/} CC_u^{s\lambda},$$

$$\frac{dJ(s)}{ds} = (J(s) \mathbin{/\!\!/} \operatorname{der}(u \mapsto [\lambda_s, u])) + \operatorname{div}_u \lambda_s,$$

and where $\operatorname{div}_u \lambda := \operatorname{tr}(u\sigma_u(\lambda))$, $\sigma_u(v) := \delta_{uv}$, $\sigma_u([\lambda_1,\lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$ and $\iota$ is the inclusion $FL \hookrightarrow FA$:



**Claim.** $CC_u^{\operatorname{bch}(\lambda_1,\lambda_2)} = CC_u^{\lambda_1} \mathbin{/\!\!/} CC_u^{\lambda_2 /\!\!/ CC_u^{\lambda_1}}$ and

$$J_u(\operatorname{bch}(\lambda_1,\lambda_2)) = J_u(\lambda_1) \mathbin{/\!\!/} CC_u^{\lambda_2 /\!\!/ CC_u^{\lambda_1}} + J_u(\lambda_2 \mathbin{/\!\!/} CC_u^{\lambda_1}),$$

and hence $tm$, $hm$, and $hta$ form a meta-group-action.

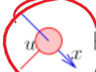
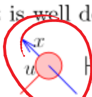
**Why ODEs?** **Q.** Find $f$ s.t. $f(x+y) = f(x)f(y)$. **A.** $\frac{df(s)}{ds} = \frac{d}{d\epsilon}f(s+\epsilon) = \frac{d}{d\epsilon}f(s)f(\epsilon) = f(s)C$. Now solve this ODE using Picard's theorem or power series.

Alekseev Torossian

**The Invariant $\zeta$.** Set $\zeta(\rho^{\pm}) = (\pm u_x, 0)$. This at least defines an invariant of u/v/w-tangles, and if the topologists will deliver a "Reidemeister" theorem, it is well defined on $\mathcal{K}^{bh}$.

orientations

$2_\varepsilon$

$\zeta$:  $\mapsto \left(+\left|{}^{u}_{x}\right., 0\right)$  $\mapsto \left(-\left|{}^{u}_{x}\right., 0\right)$

**Theorem.** $\zeta$ is (the log of) a universal finite type invariant (a homomorphic expansion) of w-tangles.

**Tensorial Interpretation.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \operatorname{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T,H) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \oplus_H \mathfrak{g})$, and hence

$$e^{\tau} : M(T,H) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

**$\zeta$ and BF Theory.** Let $A$ denote a $\mathfrak{g}$-connection on $S^4$ with curvature $F_A$, and $B$ a $\mathfrak{g}^*$-valued 2-form on $S^4$. For a hoop $\gamma_x$, let $\operatorname{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of $A$ along $\gamma_x$. For a ball $\gamma_u$, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be the integral of $B$ (transported via $A$ to $\infty$) on $\gamma_u$.

Cattaneo

**Loose Conjecture.** For $\gamma \in \mathcal{K}(T,H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B)} \bigotimes_x \operatorname{hol}_{\gamma_x}(A) = e^{\tau}(\zeta(\gamma)).$$

That is, $\zeta$ is a complete evaluation of the BF TQFT.

**Issues.** How exactly is $B$ transported via $A$ to $\infty$? How does the ribbon condition arise? Or if it doesn't, could it be that $\zeta$ can be generalized??

**The $\beta$ quotient, 1.** • Arises when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra.

• Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

**The $\beta$ quotient, 2.** Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central $R$ and with $[u,v] = c_u v - c_v u$ for $u,v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to (\bar\lambda, \omega) \quad \text{with } \bar\lambda = \sum_{x\in H,\, u\in T} \lambda_{ux} u x, \quad \lambda_{ux}, \omega \in R,$$

$$\operatorname{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v}-1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u \mathbin{/\!\!/} CC_u^{\lambda} = \left(1 + c_u\lambda_u \frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1}\left(e^{c_\lambda}u - c_u\frac{e^{c_\lambda}-1}{c_\lambda}\sum_{v\neq u}\lambda_v v\right),$$

$\operatorname{div}_u \lambda = c_u \lambda_u$, and the ODE for $J$ integrates to

$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda}c_u\lambda_u\right),$$

so $\zeta$ is formula-computable to all orders!

~~Can we simplify?~~ Repackaging.

**$\beta$ Calculus.** Let $\beta(H,T)$ be

$$\left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l} \omega \text{ and the } \alpha_{ux}\text{'s are ratio-} \\ \text{nal functions in variables} \\ t_u, \text{ one for each } u \in T. \end{array} \right\},$$

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$hta^{xu} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

**Why bother?** An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination!). If there should be an Alexander invariant to have an algebraic categorifcation, it is this one!

Pictures: Morrison, Gukov, Soltani

A. statement on long knots w is the Alexander polynomial

Further directions. v-knots? AT & KV? Jones? F-K?

[illegible] v-knots? [illegible]

Jones? E-K?

Some references.

w is the Alexander polynomial.