Hamburg bh Handout as of August 6

August-06-12
11:44 AM

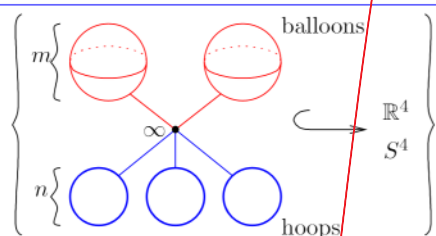
replace by: Scheme. 1. Balloons & hoops in $\mathbb{R}^4$.
2. An ansatz for an invariant: computable, related to F.T. & to BF.
3. Reduction to an "ultimate Alexander invariant.

Balloons and Hoops and their Universal Finite–Type Invariant, BF Theory, and an Ultimate Alexander Invariant

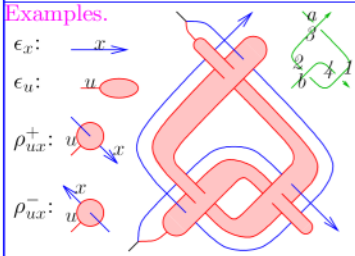
Dror Bar–Natan in Hamburg, August 2012
http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

Abstract. Balloons are two-dimensional spheres. Hoops are one dimensional loops. Knotted Balloons and Hoops (KBH) in 4-space behave much like the first and second fundamental groups of a topological space - hoops can be composed like in π_1, balloons like in π_2, and hoops "act" on balloons as π_1 acts on π_2. We will observe that ordinary knots and tangles in 3-space map into KBH in 4-space and become amalgams of both balloons and hoops.
We give an ansatz for a tree and wheel (that is, free-Lie and cyclic word) -valued invariant Z of KBHs in terms of the said compositions and action and we explain its relationship with finite type invariants. We speculate that Z is a complete evaluation of the BF topological quantum field theory in 4D, though we are not sure what that means. We show that a certain "reduction and repackaging" of Z is an "ultimate Alexander invariant" that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, here's a wonderful playground.

$\mathcal{K}^{bh}(m,n)$.



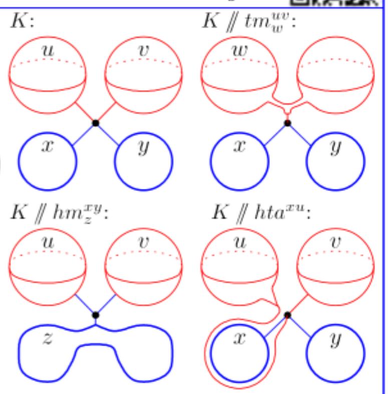
Examples.




```
T0 = Rm[3, a] Rp[b, 2] Rp[1, 4];
ζ = T0 // dm[1, 2, 1] // dm[b, 4, b] //
    dm[a, 1, a] // dm[a, 3, a];
ζ[[5]]
```

μ[CWS[0, -[ab], $-\frac{[aab]}{2}-\frac{[abb]}{2}$, $-\frac{[aaab]}{6}+\frac{7[aabb]}{4}-\frac{5[abab]}{2}-\frac{[abbb]}{6}$, $-\frac{[aaaab]}{24}+\frac{11[aaabb]}{12}-\frac{3[aabab]}{2}+\frac{11[aabbb]}{12}-\frac{3[ababb]}{2}-\frac{[abbbb]}{24}$],
h[b] LS[⟨a⟩, −⟨ab⟩, $-\langle aab\rangle+\frac{\langle abb\rangle}{2}$, $-\frac{\langle aaab\rangle}{2}+\frac{3\langle aabb\rangle}{2}-\frac{\langle abbb\rangle}{6}$, $-\frac{\langle aaaab\rangle}{6}+\frac{7\langle aaabb\rangle}{4}-\frac{7\langle aabbb\rangle}{6}-\langle ababb\rangle+\frac{\langle abbbb\rangle}{24}$],
h[a] LS[−⟨a⟩ + ⟨b⟩, $\frac{\langle ab\rangle}{2}$, $\frac{\langle aab\rangle}{12}-\frac{13\langle abb\rangle}{12}$, $-\frac{25\langle aabb\rangle}{24}+\frac{\langle abbb\rangle}{2}$, $-\frac{\langle aaaab\rangle}{720}-\frac{91\langle aaabb\rangle}{180}+\frac{59\langle aabab\rangle}{360}+\frac{143\langle aabbb\rangle}{90}+\frac{77\langle ababb\rangle}{40}-\frac{119\langle abbbb\rangle}{720}$]]

Meta-Group-Action. If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

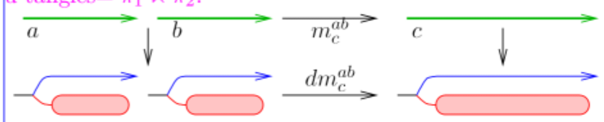


("$/\!\!/$" is newspeak for "apply an operator" and for "composition left to right")

Properties.

- Associativities: $m^{xy}_x /\!\!/ m^{xz}_x = m^{yz}_y /\!\!/ m^{xy}_x$, for $m = tm, hm$.
- Action axiom t: $tm^{uv}_w /\!\!/ hta^{xw} = hta^{xu} /\!\!/ hta^{xv} /\!\!/ tm^{uv}_w$.
- Action axiom h: $hm^{xy}_z /\!\!/ hta^{zu} = hta^{xu} /\!\!/ hta^{yu} /\!\!/ hm^{xy}_z$.
- SD Product: $dm^{ab}_c := hta^{ba} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$ is associative.

u-tangles$=\pi_1 \ltimes \pi_2$.



The Meta-Group-Action M. Let T be a set of "tail labels" ("balloon colours"), H a set of "head labels" ("hoop colours"), and let $\{h_x\}_{x\in H}$ be a set of formal symbols. Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on T, so there's tr : $FA \to CW$, and both FA and CW are FL-modules. Let

$$M(T,H) := \{\mu = (\omega, \bar\lambda = (\lambda_x)_{x\in H}) :\ \omega \in CW,\ \lambda_x \in FL\}$$

(e.g., $\mu = \ldots$).
With $\mu = (\omega, \bar\lambda)$ define

$$tm^{uv}_w : \mu \mapsto \mu /\!\!/ (u, v \mapsto w),$$

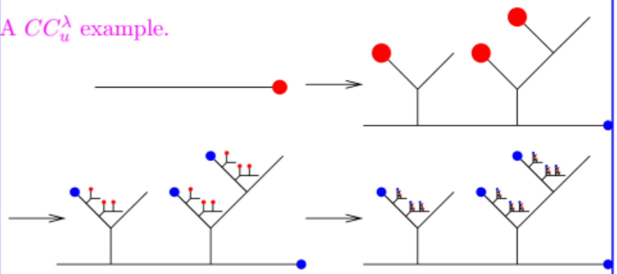
$$hm^{xy}_z : \mu \mapsto \left(\omega, \left(\ldots, \widehat{\lambda_x}, \widehat{\lambda_y}, \ldots, \mathrm{bch}(\lambda_x, \lambda_y)_z\right)\right)$$

$$hta^{xu} : \mu \mapsto \underbrace{\mu \overset{\text{"stable apply"}}{/\!\!/\!\!/} (u \mapsto e^{\mathrm{ad}\,\lambda_x}(\bar u)) /\!\!/ (\bar u \mapsto u)}_{\mu /\!\!/ CC^{\lambda_x}_u} + \underbrace{(J_u(\lambda_x), 0)}_{\text{the "}J\text{-spice"}}$$

A CC^{λ}_u example.

$$hta^{xu} : \mu \mapsto \underbrace{\mu \;/\!/\!/\; (u \mapsto e^{\operatorname{ad}\lambda_x}(\bar u)) \;/\!/\; (\bar u \mapsto u)}_{\mu /\!/ CC_u^{\lambda_x}} + \underbrace{(J_u(\lambda_x), 0)}_{\text{the "}J\text{-spice"}}$$

A CC_u^λ example.



Teichner by Bergman

- u-Knots inject into $\mathcal{K}^{bh}$ (likely u-tangles too).
- In fact, n-component v/w-tangles map into $\mathcal{K}^{bh}(n,n)$ and we have a conjectural understanding of $\mathcal{K}^{bh}$ in these terms.

C ✓

C: It would be nice to have a precise statement. ✓

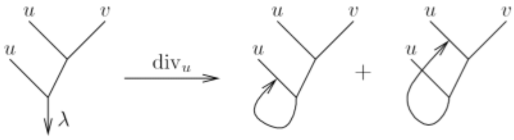
Balloons and Hoops and their Universal Finite–Type Invariant, 2

The Meta-Cocycle J. Set $J_u(\lambda) := J(1)$ where

$$J(0)=0, \qquad \lambda_s = \lambda \mathbin{/\!\!/} CC_u^{s\lambda},$$

$$\frac{dJ(s)}{ds} = (J(s) \mathbin{/\!\!/} \operatorname{der}(u \mapsto [\lambda_s, u])) + \overset{B}{\operatorname{div}_u} \lambda_s,$$

and where $\operatorname{div}_u \lambda := \operatorname{tr}(u\sigma_u(\lambda))$, $\sigma_u(v) := \delta_{uv}$, $\sigma_u([\lambda_1,\lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$ and ι is the inclusion $FL \hookrightarrow FA$:



Claim.

$$CC_u^{\operatorname{bch}(\lambda_1,\lambda_2)} = CC_u^{\lambda_1} \mathbin{/\!\!/} CC_u^{\lambda_2 /\!\!/ CC_u^{\lambda_1}},$$

$$J_u(\operatorname{bch}(\lambda_1,\lambda_2)) = J_u(\lambda_1) \mathbin{/\!\!/} CC_u^{\lambda_2 /\!\!/ CC_u^{\lambda_1}} + J_u(\lambda_2 \mathbin{/\!\!/} CC_u^{\lambda_1}),$$

and hence tm, hm, and hta form a meta-group-action.

A note about e^x: $\boxed{\text{Alex}}$ $\boxed{\text{Tor}}$

The invariant ζ.

need an aside on how FL/CW parametrize formulas in f.d. Lie algebras

Also, "the Alexander relation".

The β quotient. Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u,v \in T$. Then $FL \to L_\beta$ and $CW \overset{A}{\to} R$. Under this,

$$\mu \to (\omega, \bar\lambda) \quad \text{with } \omega \in R, \quad \bar\lambda = \sum_{x\in H,\, u\in T} \lambda_{ux} ux,$$

$$\operatorname{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v}-1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u \mathbin{/\!\!/} CC_u^\lambda = \left(1 + c_u\lambda_u \frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1}\left(e^{c_\lambda}u - c_u\frac{e^{c_\lambda}-1}{c_\lambda}\sum_{v\neq u}\lambda_v v\right),$$

$\operatorname{div}_u \lambda = c_u\lambda_u$, and the ODE for J integrates to

$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda}c_u\lambda_u\right),$$

so ζ is formula-computable to all orders! Can we simplify?

Repackaging ✓

β Calculus. Let $\beta(H,T)$ be

$$\left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l} \omega \text{ and the } \alpha_{ux}\text{'s are ratio-} \\ \text{nal functions in variables} \\ t_u, \text{ one for each } u \in T. \end{array} \right\},$$

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$hta^{xu} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

Polynomiality, efficiency, categorification

Gukov Morrison

Further directions.

V-knots? AT & KV?

Jones? E-K?

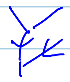
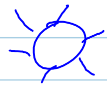
"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)
www.katlas.org

A: Is this because CW is "the coinvariants of FL"?

B. Is div the only "meta-Lie-cocycle" on M^0?

1.B. Is div the only "meta-Lie-cocycle" on M^u?

Add:

$\mathrm{Lie}_n =$  $\mathrm{tr}_n =$ 

Claim: ζ (the log of) is a "universal finite type invariant"

ζ and BF

Cattaneo

These questions remain unanswered!