Hamburg bh Handout as of August 2

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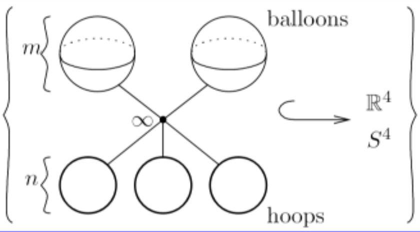
Balloons and Hoops and their Universal Finite–Type Invariant, BF Theory, and an Ultimate Alexander Invariant

Dror Bar–Natan in Hamburg, August 2012
http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

Abstract. Balloons are two-dimensional spheres. Hoops are one dimensional loops. Knotted Balloons and Hoops (KBH) in 4-space behave much like the first and second fundamental groups of a topological space - hoops can be composed like in π_1, balloons like in π_2, and hoops "act" on balloons as π_1 acts on π_2. We will observe that ordinary knots and tangles in 3-space map into KBH in 4-space and become amalgams of both balloons and hoops.
We give an ansatz for a tree and wheel (that is, free-Lie and cyclic word) -valued invariant Z of KBHs in terms of the said compositions and action and we explain its relationship with finite type invariants. We speculate that Z is a complete evaluation of the BF topological quantum field theory in 4D, though we are not sure what that means. We show that a certain "reduction and repackaging" of Z is an "ultimate Alexander invariant" that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, here's a wonderful playground.

$\mathcal{K}^{bh}(m,n)$.



Example.

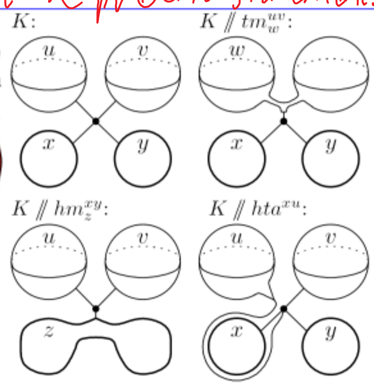
use band notation

knotted tubes $\to$ K^{bh}

Teichner by Bergman

- u-Tangles inject into $\mathcal{K}^{bh}$.
- In fact, n-component v/w-tangles map into $\mathcal{K}^{bh}(n,n)$ and we have a conjectural understanding of $\mathcal{K}^{bh}$ in these terms. It would be nice to have a precise statement.

Meta-Group-Action. If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.



("$/\!\!/$" is newspeak for "apply an operator" and for "composition left to right")

Properties.

- Associativities: $m^{xy}_x /\!\!/ m^{xz}_z = m^{yz}_y /\!\!/ m^{xy}_x$, for $m = tm, hm$.
- Action axiom t: $tm^{uv}_w /\!\!/ hta^{xw} = hta^{xu} /\!\!/ hta^{xv} /\!\!/ tm^{uv}_w$,
- Action axiom h: $hm^{xy}_z /\!\!/ hta^{zu} = hta^{xu}/hta^{yu} /\!\!/ hm^{xy}_z$.
- SD Product: $dm^{ab}_c := hta^{ba} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$ is associative.

The Meta-Group-Action M. Let T be a set of "tail labels" ("balloon colours"), H a set of "head labels" ("hoop colours"), and let $\{h_x\}_{x\in H}$ be a set of formal symbols. Let $FL = FL(T)$ and $FA = FA(T)$ be the (graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (graded) vector space of cyclic words on T, so there's $\operatorname{tr}: FA \to CW$, and both FA and CW are FL-modules. Let

completed

completed

$$M(T,H) := \{\mu = (\omega, \lambda = (\lambda_x)_{x\in H}) :\ \omega \in CW,\ \lambda_x \in FL\}$$

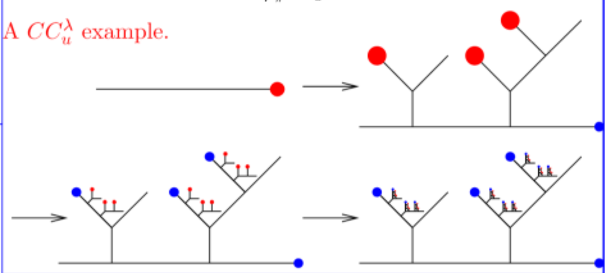
(e.g., $\mu = \ldots$).
With $\mu = (\omega, \lambda)$ define

$$tm^{uv}_w : \mu \mapsto \mu /\!\!/ (u, v \mapsto w),$$

$$hm^{xy}_z : \mu \mapsto \left(\omega, \left(\ldots, \widehat{\lambda_x}, \widehat{\lambda_y}, \ldots, \operatorname{bch}(\lambda_x,\lambda_y)_z\right)\right)$$

$$hta^{xu} : \mu \mapsto \underbrace{\mu \overbrace{/\!\!/\!\!/}^{\text{"stable apply"}} (u \mapsto e^{\operatorname{ad}\lambda_x}(\bar u)) /\!\!/ (\bar u \mapsto u)}_{\mu /\!\!/ CC_u^{\lambda_x}} + \underbrace{(J_u(\lambda_x), 0)}_{\text{the "}J\text{-spice"}}$$

A CC^λ_u example.



The Meta-Cocycle J. Set $J_u(\lambda) := J(1)$ where

$$J(0) = 0, \qquad \lambda_s = \lambda /\!\!/ CC^{s\lambda}_u,$$

$$\frac{dJ(s)}{ds} = (J(s) /\!\!/ \operatorname{der}(u \mapsto [\lambda_s, u])) + \operatorname{div}_u \lambda_s.$$

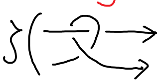
Claim.

$$CC^{\operatorname{bch}(\lambda_1,\lambda_2)}_u = CC^{\lambda_1}_u /\!\!/ CC^{\lambda_2 /\!\!/ CC^{\lambda_1}_u}_u,$$

$$J_u(\operatorname{bch}(\lambda_1,\lambda_2)) = J_u(\lambda_1) /\!\!/ CC^{\lambda_2 /\!\!/ CC^{\lambda_1}_u}_u + J_u(\lambda_2 /\!\!/ CC^{\lambda_1}_u),$$

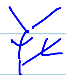
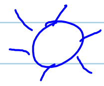
and hence tm, hm, and hta form a meta-group-action.

The invariant Z

Example: Z()

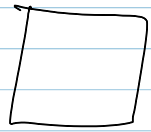
Add:

(The log of) Claim. Z is a "universal

$\mathrm{Lie}_n =$  $\mathrm{tr}_n =$ 

Claim. ζ is a "universal finite type invariant"

ζ and BF



Cattaneo

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need an aside on how FL/CW parametrize formulas in f.d. Lie algebras.

Also, say "the Alexander relation"

The β quotient. Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u,v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to (\omega, \bar\lambda) \quad \text{with } \omega \in R, \quad \bar\lambda = \sum_{x\in H,\, u\in T} \lambda_{ux} ux,$$

$$\mathrm{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v} - 1}\left(\frac{e^{c_u}-1}{c_u}u + e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u /\!/ CC_u^\lambda = \left(1 + c_u\lambda_u\frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1}\left(e^{c_\lambda}u - c_u\frac{e^{c_\lambda}-1}{c_\lambda}\sum_{v\neq u}\lambda_v v\right),$$

$\mathrm{div}_u\,\lambda = c_u\lambda_u$, and the ODE for J integrates to

$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda}c_u\lambda_u\right),$$

so ζ is formula-computable exactly to all orders!

ζ/A, repackaged

Polynomiality, efficiency, categorification

Gukov

Morrison

Further directions.

v-knots?

Jones?

AT & KV?

E-K?

must put pictures of A&T somewhere.

Somewhere:

Proof. 1. I can prove.

2. Who needs a proof when you can compute?

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas