Hamburg KH Handout as of July 23

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~~A Quick Reference Guide to Khovanov's Categorification of the Jones Polynomial~~

Dror Bar-Natan, July 23, 2012

The Kauffman Bracket: $\langle\emptyset\rangle = 1$; $\langle\bigcirc L\rangle = (q+q^{-1})\langle L\rangle$; $\langle\times\rangle = \langle\underset{0-\text{smoothing}}{\asymp}\rangle - q\langle\underset{1-\text{smoothing}}{)(}\rangle$.
The Jones Polynomial: $\hat{J}(L) = (-1)^{n_-}q^{n_+-2n_-}\langle L\rangle$, where (n_+, n_-) count $(\times, \times)$ crossings.

modernize ✓

Khovanov's construction: $[\![L]\!]$ — a chain complex of graded $\mathbb{Z}$-modules;

$$[\![\emptyset]\!] = 0 \to \underset{\text{height }0}{\mathbb{Z}} \to 0; \qquad [\![\bigcirc L]\!] = V\otimes[\![L]\!]; \qquad [\![\times]\!] = \text{Flatten}\left(0 \to \underset{\text{height }0}{[\![\asymp]\!]} \to \underset{\text{height }1}{[\![)(]\!]\{1\}} \to 0\right);$$

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$$\mathcal{H}(L) = \mathcal{H}\left(\mathcal{C}(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}\right)$$

$$V = \text{span}\langle v_+, v_-\rangle; \qquad \deg v_\pm = \pm 1; \qquad q\dim V = q + q^{-1} \quad \text{with} \quad q\dim\mathcal{O} := \sum_m q^m \dim \mathcal{O}_m;$$

$$\mathcal{O}\{l\}_m := \mathcal{O}_{m-l} \quad \text{so} \quad q\dim\mathcal{O}\{l\} = q^l\, q\dim\mathcal{O}; \qquad \cdot[s]: \quad \text{height shift by } s;$$

$$(\bigcirc\ \bigcirc \longrightarrow \bigcirc\!\!\bigcirc) \longrightarrow (V\otimes V \xrightarrow{m} V) \qquad m: \begin{cases} v_+\otimes v_- \mapsto v_- & v_+\otimes v_+ \mapsto v_+ \\ v_-\otimes v_+ \mapsto v_- & v_-\otimes v_- \mapsto 0 \end{cases}$$

$$(\bigcirc\!\!\bigcirc \longrightarrow \bigcirc\ \bigcirc) \longrightarrow (V \xrightarrow{\Delta} V\otimes V) \qquad \Delta: \begin{cases} v_+ \mapsto v_+\otimes v_- + v_-\otimes v_+ \\ v_- \mapsto v_-\otimes v_- \end{cases}$$

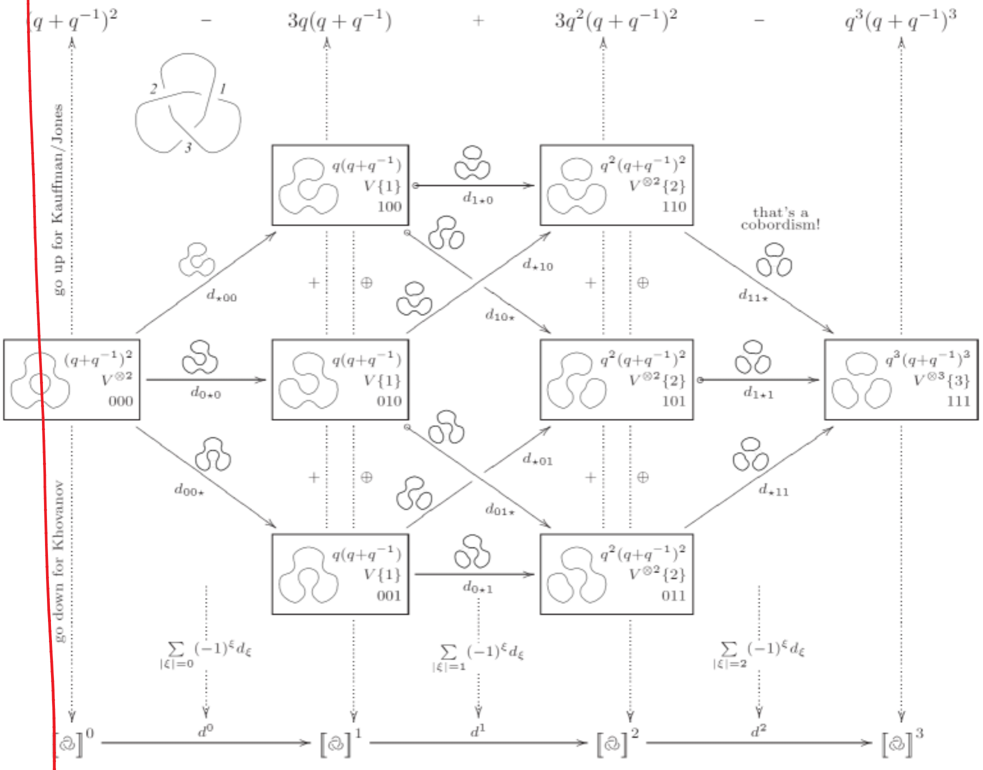
That's a Frobenius Algebra! And a (1+1)-dimensional TQFT!

Example:

$$q^{-2} + 1 + q^2 - q^6 \xrightarrow[\text{(with }(n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q + q^3 + q^5 - q^9.$$

$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3$$

philosophy column...



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $\qquad = \quad [\![\circledast]\!] \xrightarrow[\text{(with }(n_+,n_-)=(3,0))]{\cdot[-n_-]\{n_+-2n_-\}} \mathcal{C}(\circledast).$

Theorem 1. The graded Euler characteristic of $\mathcal{C}(L)$ is $\hat{J}(L)$.
Theorem 2. The homology $\mathcal{H}(L)$ is a link invariant and thus so is $\text{Kh}_{\mathbb{F}}(L) := \sum_r t^r\, q\dim\mathcal{H}^r_{\mathbb{F}}(\mathcal{C}(L))$ over any field $\mathbb{F}$.
Theorem 3. $\mathcal{H}(\mathcal{C}(L))$ is strictly stronger than $\hat{J}(L)$: $\mathcal{H}(\mathcal{C}(\bar{5}_1)) \neq \mathcal{H}(\mathcal{C}(10_{132}))$ whereas $\hat{J}(\bar{5}_1) = \hat{J}(10_{132})$.
~~**Conjecture 1.** $\text{Kh}_{\mathbb{Q}}(L) = q^{s-1}(1+q^2+(1+tq^4)\text{Kh}')$ and $\text{Kh}_{\mathbb{F}_2}(L) = q^{s-1}(1+q^2)(1+(1+tq^2)\text{Kh}')$ for even $s = s(L)$ and non-negative coefficients laurent polynomial $\text{Kh}' = \text{Kh}'(L)$.~~
Conjecture 2. For alternating knots s is the signature and Kh' depends only on tq^2.
References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and DBN's
`http://www.ma.huji.ac.il/~drorbn/papers/Categorification/`

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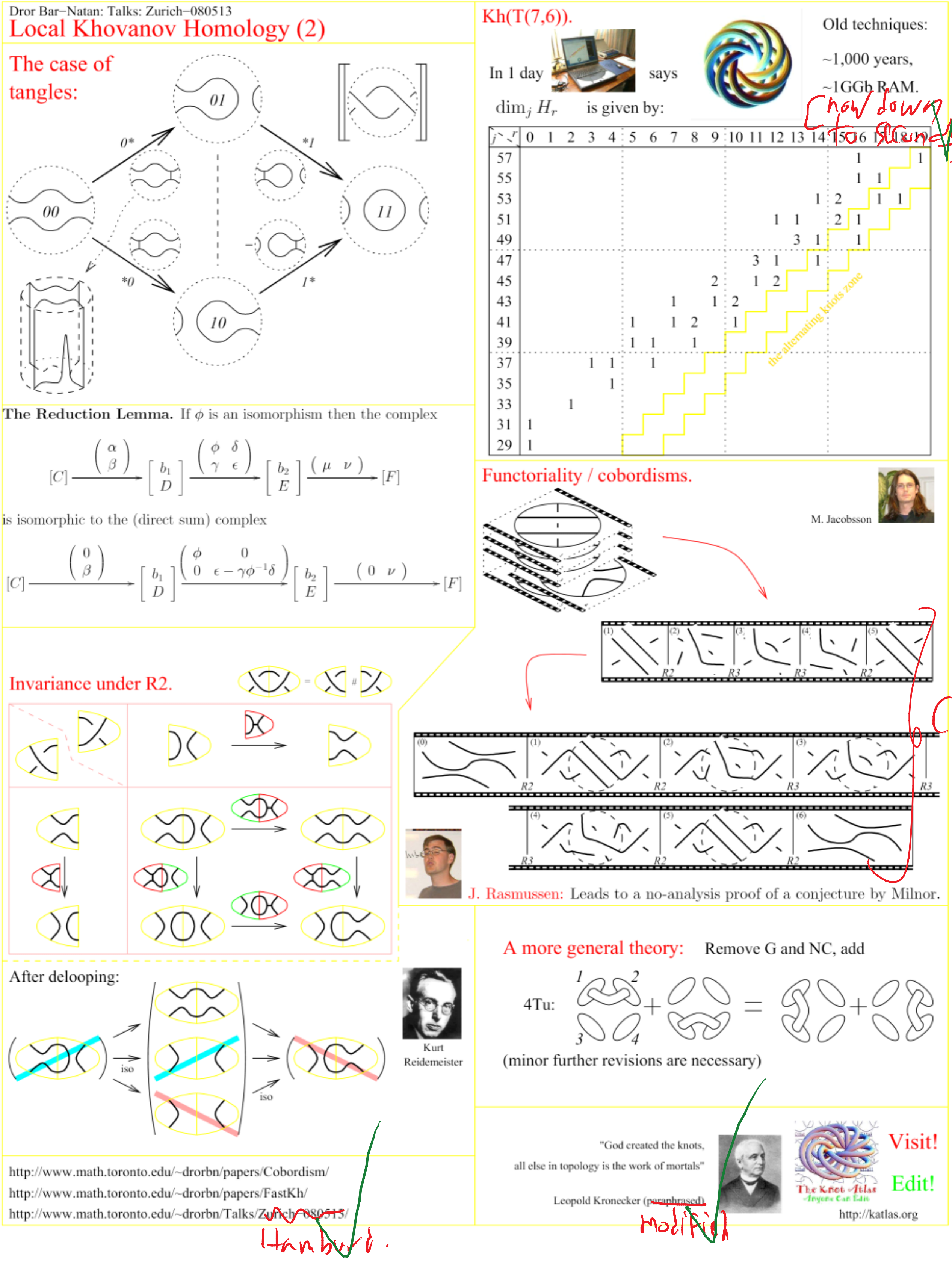
toronto



Dror Bar–Natan: Talks: ~~Zurich–080513~~ Hamburg.

Local Khovanov Homology (2)

The case of tangles:

The Reduction Lemma. If ϕ is an isomorphism then the complex

$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta \\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0 \\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

Invariance under R2.

After delooping:

http://www.math.toronto.edu/~drorbn/papers/Cobordism/
http://www.math.toronto.edu/~drorbn/papers/FastKh/
http://www.math.toronto.edu/~drorbn/Talks/~~Zurich–080513~~/

Kh(T(7,6)).

In 1 day says $\dim_j H_r$ is given by:

Old techniques:
~1,000 years,
~1GGb RAM.

[now down to seconds)

$j \setminus r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

Functoriality / cobordisms.

clean

J. Rasmussen: Leads to a no-analysis proof of a conjecture by Milnor.

A more general theory: Remove G and NC, add

4Tu:

(minor further revisions are necessary)

"God created the knots,
all else in topology is the work of mortals"

Leopold Kronecker (~~paraphrased~~) modified

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http://katlas.org

Consider a story around:

From http://www.webcrafterz.net/table/images/MonopolyBoard2.jpg

From http://www.onemoretap.com/2009/02/27/review-monopoly-world-edition/ :

Oh no! Taxicab has been sent to jail.

From http://cheat-pcgame.blogspot.ca/2011/06/download-game-monopoly-here-now-full.html :