Hamburg KH Handout as of August 23

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With $\omega\epsilon\beta$:= http://www.math.toronto.edu/drorbn/Talks/Hamburg-1208

A Quick Introduction to Khovanov Homology

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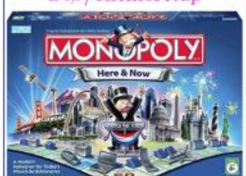
Hamburg, August 2012

Abstract. I will tell the Kauffman bracket story of the Jones polynomial as Kauffman told it in 1987, then the Khovanov homology story as Khovanov told it in 1999, and finally the "local Khovanov homology" story as I understood it in 2003. At the end of our 90 minutes we will understand what is a "Jones homology", how to generalize it to tangles and to cobordisms between tangles, and why it is computable relatively efficiently. But we will say nothing about more modern stuff — the Rasmussen invariant, Alexander and HOMFLYPT knot homologies, and the categorification of sl_2 and other Lie algebras.

Why Bother?

$\omega\epsilon\beta$/webcrafterz

$\omega\epsilon\beta$/onemoretap



$\omega\epsilon\beta$/cheat-pcgame

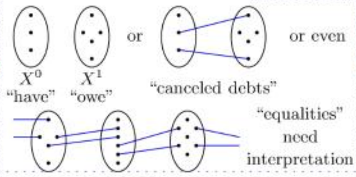
What is Categorification=Concretization=de-abstraction? "3" is {cow, cow, cow} and {pig, pig, pig} and many other things…

…categorification is choosing which 3 it is!

$\mathbb{N}$. Natural numbers $\mapsto$ finite sets, equalities $\mapsto$ bijections, inequalities $\mapsto$ injections and surjections:

$$\binom{2n}{n}=\sum\binom{n}{k}^2 \mapsto \binom{X\times\{1,2\}}{|X|}\leftrightarrow\bigcup\binom{X}{k}\times\binom{X}{k}$$

$\mathbb{Z}$. Negative numbers:



The Philosophy Corner

Weaker Categorification. Do the same in the category of vector spaces: "3" becomes V s.t. $\dim V=3$, or better, $V^\bullet=(\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$ and $\chi(V^\bullet):=\sum(-1)^r\dim V^r=3=\sum(-1)^r\dim H^r$. Equalities become homotopies between complexes.

Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules; $V=\operatorname{span}\langle v_+,v_-\rangle$; $\deg v_\pm=\pm1$; $q\dim V=q+q^{-1}$;

$$K(\bigcirc^k)=V^{\otimes k};\quad K(\times)=\text{Flatten}\Big(0\to \underset{\text{height }0}{K(\asymp)\{1\}}\to \underset{\text{height }1}{K()()\{2\}}\to 0\Big);$$

$$K(\times)=\text{Flatten}\Big(0\to \underset{\text{height }-1}{K()()\{-2\}}\to \underset{\text{height }0}{K(\asymp)\{-1\}}\to 0\Big);$$

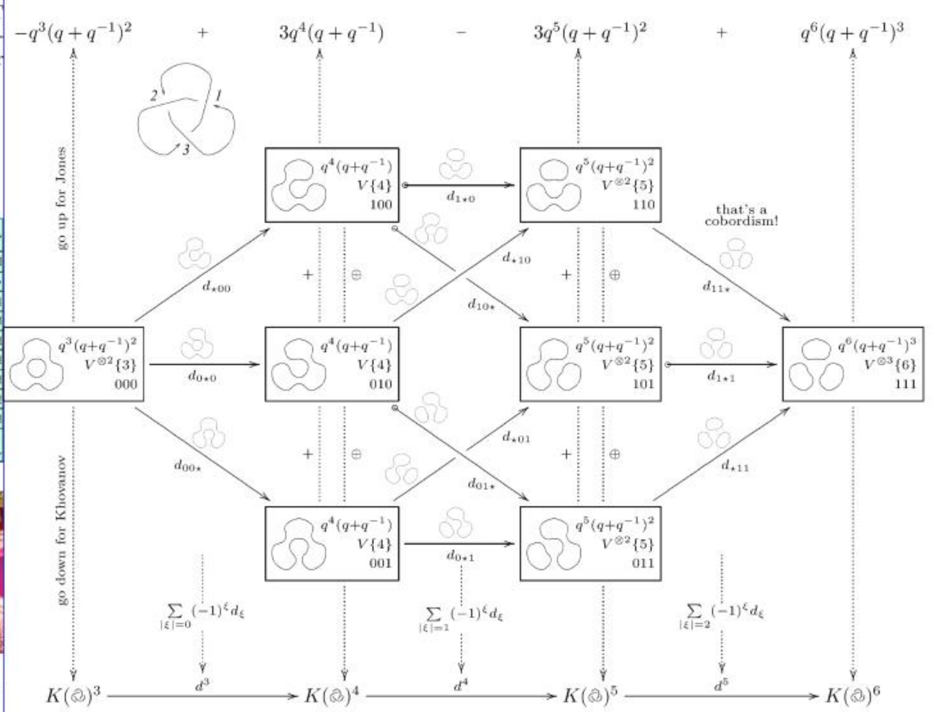
Categorifying $\mathbb{Z}[q^{\pm1}]$. $f=\sum a_jq^j$ becomes $V=\bigoplus V_j$ s.t. $q\dim V:=\sum q^j\dim V_j=f$, or better, $V^\bullet=(\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$, $\deg d=0$, and $\chi_q(V^\bullet):=\sum(-1)^r q\dim V^r=f=\sum(-1)^r q\dim H^r$.

Note. Setting $V\{l\}_j:=V_{j-l}$, we get $q\dim V\{l\}=q^l q\dim V$.

$$(\bigcirc\bigcirc \to \bigcirc\!\!\bigcirc)\longrightarrow(V\otimes V\xrightarrow{m}V)\qquad m:\begin{cases}v_+\otimes v_-\mapsto v_- & v_+\otimes v_+\mapsto v_+\\ v_-\otimes v_+\mapsto v_- & v_-\otimes v_-\mapsto 0\end{cases}$$

$$(\bigcirc\!\!\bigcirc \to \bigcirc\bigcirc)\longrightarrow(V\xrightarrow{\Delta}V\otimes V)\qquad \Delta:\begin{cases}v_+\mapsto v_+\otimes v_-+v_-\otimes v_+\\ v_-\mapsto v_-\otimes v_-\end{cases}$$

Example:



(here $(-1)^\xi:=(-1)^{\sum_{i<j}\xi_i}$ if $\xi_j=\star$)

Theorem 1. The graded Euler characteristic of $K(L)$ is $J(L)$.

Theorem 2. The homology $\mathrm{Kh}(L)$ of $K(L)$ is a link invariant.

Theorem 3. $\mathrm{Kh}(L)$ is strictly stronger than $J(L)$: $\mathrm{Kh}(\bar 5_1)\neq\mathrm{Kh}(10_{132})$ whereas $J(\bar 5_1)=J(10_{132})$.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and my http://www.math.toronto.edu/~drorbn/papers/Categorification/.

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Local Khovanov Homology (1)

(an outdated overview)

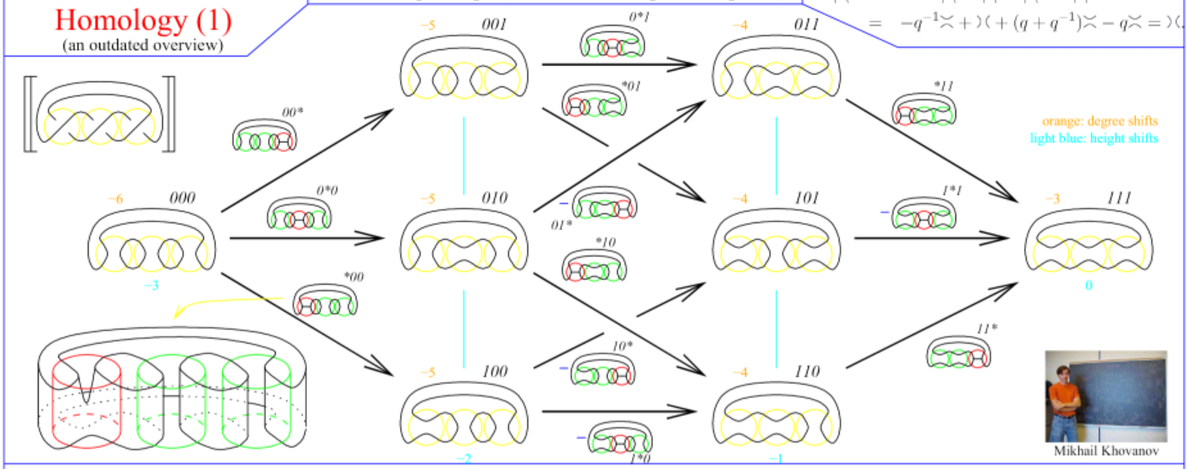
The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$

$J: \times \mapsto q)(- q^2 \asymp, \quad J: \times \mapsto -q^{-2}\asymp + q^{-1})($

R2

$J: \mapsto -q^{-1} + + - q$

$= -q^{-1}\asymp +)(+ (q+q^{-1})\asymp - q\asymp =)($



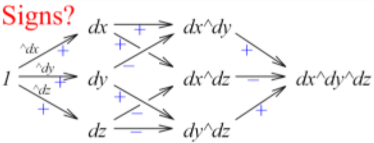
What is it?

A cube for each knot/link projection;

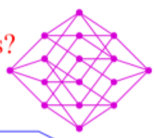
Vertices: All fillings of  with  or with .

Edges: All fillings of $I\times$  with $I\times$  or with $I\times$  and precisely one .

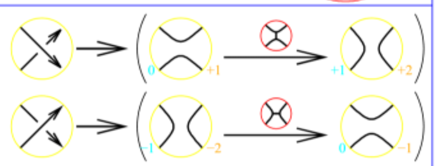
Signs?



More crossings?



General Crossings



Where does it live?

In *Kom(Mat(<Cob> / {S, T, G, NC})) / homotopy*

Kom: Complexes Mat: Matrices

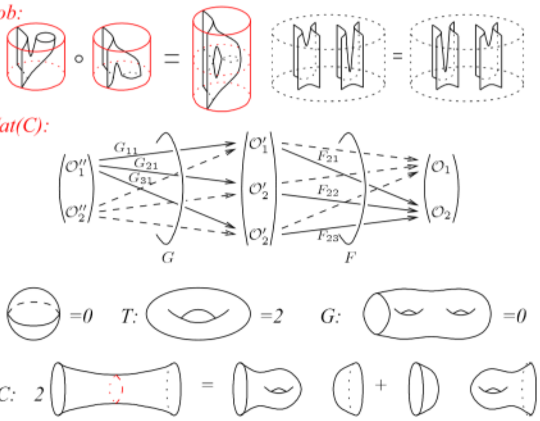
Cob: Cobordisms <...>: Formal lin. comb.

Cob:

Mat(C):

S: =0 *T:* =2 *G:* =0

NC: 2 = +



Complexes:

$$\Omega = \left(\Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \longrightarrow \Omega^{n_+}\right)$$

Morphisms:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \longrightarrow \cdots \\ & \downarrow F^{r-1} & & \downarrow F^r & & \downarrow F^{r+1} & \\ \cdots \longrightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \longrightarrow \cdots \end{array}$$

Homotopies:

$$\begin{array}{ccccc} \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} \\ F^{r-1}\Downarrow G^{r-1} & h^r \swarrow & F^r \Downarrow G^r & h^{r+1} \swarrow & F^{r+1}\Downarrow G^{r+1} \\ \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} \end{array}$$

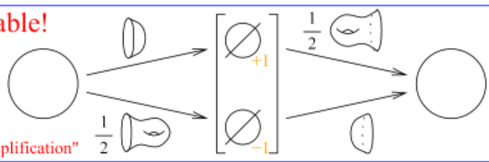
$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

All arrows in an arbitrary additive category!

The Main Point. "The cube", Kh(L), is an up-to-homotopy invariant of knots and links. It's Euler characteristic is the Jones polynomial, yet it is strictly stronger than the Jones polynomial. It is functorial (in the appropriate sense) and practically computable.

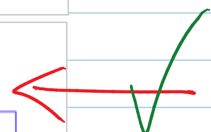
Computable!

via "complex simplification"



The Categorification Speculative Paradigm.
- Every object in math is the Euler characteristic of a complex.
- Every operation lifts to an operation between complexes.
- Every identity remains true, up to homotopy.

I mean business



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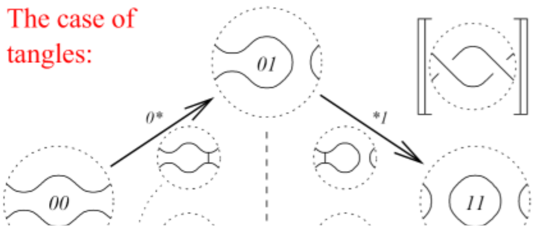
Local Khovanov Homology (2)

The case of tangles:



Kh(T(7,6)).

In 1 day  says $\dim_j H_r$ is given by:

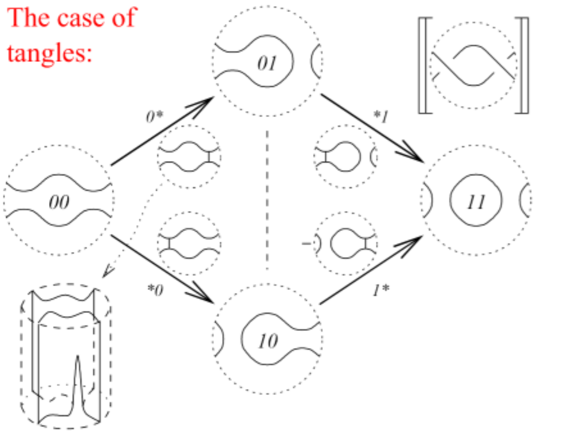
Old techniques:
~1,000 years,
~1GGb RAM.
(now down to seconds)

j \ r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			

I mean business

Local Khovanov Homology (2)

The case of tangles:



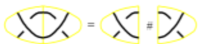
The Reduction Lemma. If ϕ is an isomorphism then the complex

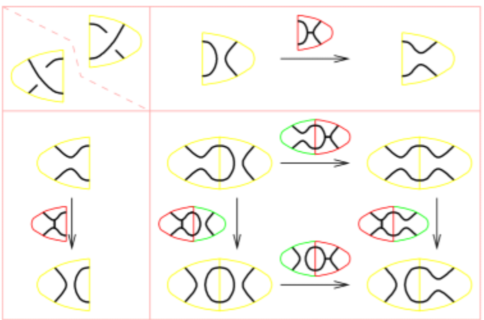
$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

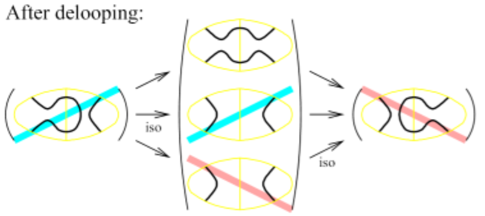
$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

Invariance under R2.





After delooping:



Kurt Reidemeister

Kh(T(7,6)).

In 1 day  says  $\dim_j H_r$ is given by:

Old techniques:
~1,000 years,
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(now down to seconds)

$j \backslash r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
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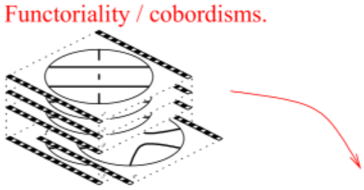
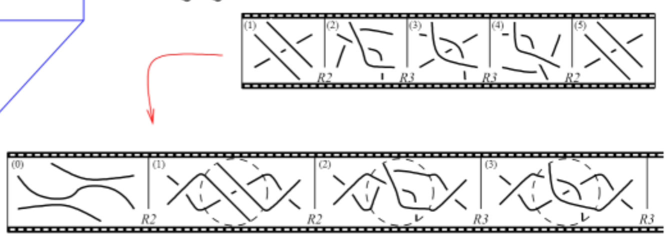
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

the alternating knots zone

Functoriality / cobordisms.



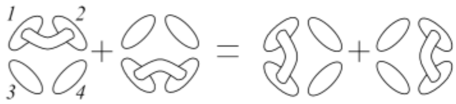
M. Jacobsson



J. Rasmussen: Leads to a no-analysis proof of a conjecture by Milnor.

A more general theory:

Remove G and NC, add

4Tu: 

(minor further revisions are necessary)

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (modified)

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http://www.math.toronto.edu/~drorbn/papers/FastKh/
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