Hamburg KH Handout as of August 12

August-12-12
4:02 PM

With $\omega\epsilon\beta$:= http://www.math.toronto.edu/drorbn/Talks/Hamburg-1208

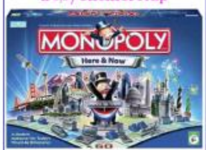
A Quick Introduction to Khovanov Homology

Dror Bar-Natan,
Hamburg, August 2012

Abstract. I will tell the Kauffman bracket story of the Jones polynomial as Kauffman told it in 1987, then the Khovanov homology story as Khovanov told it in 1999, and finally the "local Khovanov homology" story as I understood it in 2003. At the end of our 90 minutes we will understand what is a "Jones homology", how to generalize it to tangles and to cobordisms between tangles, and why it is computable relatively efficiently. But we will say nothing about more modern stuff — the Rasmussen invariant, Alexander and HOMFLYPT knot homologies, and the categorification of sl_2 and other Lie algebras.

$\omega\epsilon\beta$/webcrafterz

$\omega\epsilon\beta$/onemoretap



$\omega\epsilon\beta$/cheat-pcgame

A Quick Reference Guide to Khovanov's Categorification of the Jones Polynomial

The Jones Polynomal: $J(\times) = qJ(\underset{0\text{-smoothing}}{)(}) - q^2J(\underset{1\text{-smoothing}}{\asymp})$ $\quad J(\times) = -q^{-2}J(\underset{0\text{-smoothing}}{\asymp}) + q^{-1}J(\underset{1\text{-smoothing}}{)(})$;

$J(\emptyset)=1;\quad J(\bigcirc L) = (q+q^{-1})J(L)$

Some quick summary of categorification should come here: numbers, v.s., polys, graded v.s., Euler characteristics.

Khovanov's construction: $K(L)$ — a chain complex of graded $\mathbb{Z}$-modules;

$$K(\times) = \text{Flatten}\left(0 \to \underset{\text{height }0}{K()()\{1\}} \to \underset{\text{height }1}{K(\asymp)\{2\}} \to 0\right);\quad K(\times) = \text{Flatten}\left(0 \to \underset{\text{height }-1}{K(\asymp)\{-2\}} \to \underset{\text{height }0}{K()()\{-1\}} \to 0\right);$$

$$K(\emptyset) = 0 \to \underset{\text{height }0}{\mathbb{Z}} \to 0;\qquad K(\bigcirc L) = V\otimes K(L);\qquad \text{Kh}(L) = \mathcal{H}(K(L))$$

$$V = \text{span}\langle v_+, v_-\rangle;\qquad \deg v_\pm = \pm 1;\qquad q\dim V = q+q^{-1}\quad\text{with}\quad q\dim\mathcal{O} := \sum_m q^m\dim\mathcal{O}_m;$$

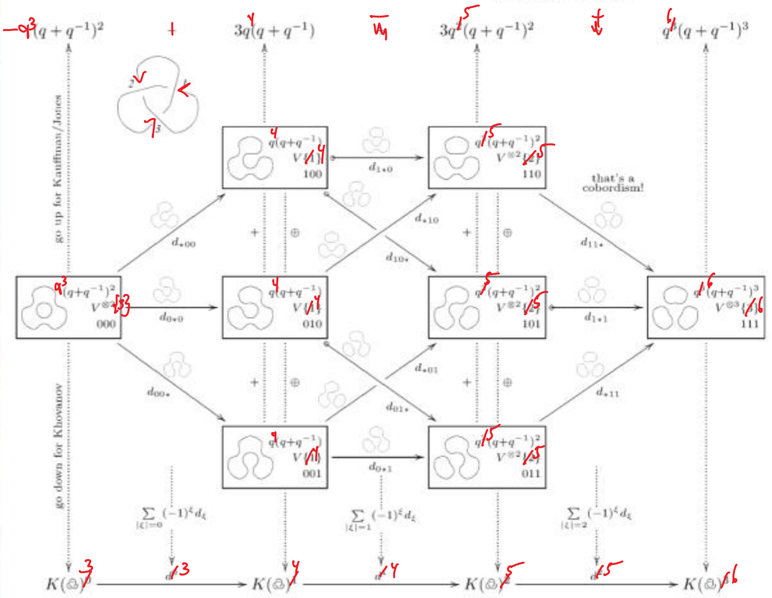
$$\mathcal{O}\{l\}_m := \mathcal{O}_{m-l}\quad\text{so}\quad q\dim\mathcal{O}\{l\} = q^l\, q\dim\mathcal{O};$$

$$(\bigcirc\bigcirc \longrightarrow \bigcirc\!\!\bigcirc) \longrightarrow (V\otimes V \xrightarrow{m} V)\qquad m: \begin{cases} v_+\otimes v_- \mapsto v_- & v_+\otimes v_+ \mapsto v_+ \\ v_-\otimes v_+ \mapsto v_- & v_-\otimes v_- \mapsto 0 \end{cases}$$

$$(\bigcirc\!\!\bigcirc \longrightarrow \bigcirc\bigcirc) \longrightarrow (V \xrightarrow{\Delta} V\otimes V)\qquad \Delta: \begin{cases} v_+ \mapsto v_+\otimes v_- + v_-\otimes v_+ \\ v_- \mapsto v_-\otimes v_- \end{cases}$$

That's a Frobenius Algebra! And a (1+1)-dimensional TQFT!

Example: $q^{-2}+1+q^2-q$ ~~$\frac{(-1)^{n_-}q^{n_+-2n_-}}{(\text{with }(n_+,n_-)=(3,0))}$~~ $q+q^3+q^5-q^9$.



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $\qquad = K(\cdot)$ ~~$[-n_-]\{n_+-2n_-\}$ (with $(n_+,n_-)=(3,0)$)~~ $\mathcal{C}(\cdot)$.

Theorem 1. The graded Euler characteristic of $K(L)$ is $J(L)$.

Theorem 2. The homology $\mathcal{H}(L)$ Kh is a link invariant. ~~and thus so is $\text{Kh}_{\mathbb{F}}(L) := \sum_r t^r q\dim\mathcal{H}_{\mathbb{F}}^r(\mathcal{C}(L))$ over any field $\mathbb{F}$.~~

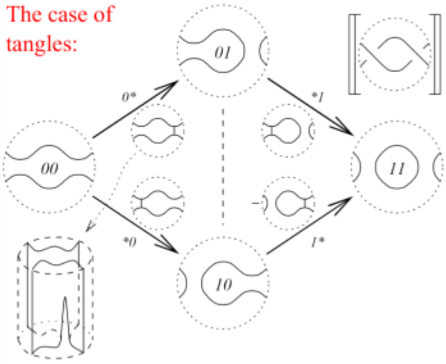
Theorem 3. ~~$\mathcal{H}(\mathcal{C}(L))$~~ Kh is strictly stronger than $J(L)$: ~~$\mathcal{H}(\mathcal{C}$~~ Kh$(\bar 5_1)) \neq$ ~~$\mathcal{H}(\mathcal{C}$~~ Kh$(10_{132}))$ whereas $J(\bar 5_1) = J(10_{132})$.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and my http://www.math.toronto.edu/~drorbn/papers/Categorification/.

Dror Bar-Natan: Talks: Hamburg-1208

Local Khovanov Homology (2)

The case of tangles:



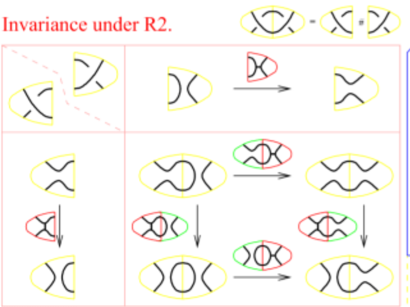
The Reduction Lemma. If ϕ is an isomorphism then the complex

$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

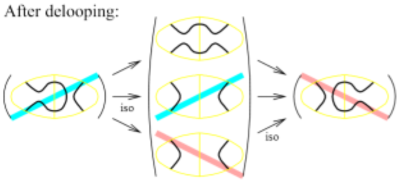
is isomorphic to the (direct sum) complex

$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

Invariance under R2.



After delooping:



Kurt Reidemeister

http://www.math.toronto.edu/~drorbn/papers/Cobordism/
http://www.math.toronto.edu/~drorbn/papers/FastKh/
http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

Kh(T(7,6)).

In 1 day  says 

Old techniques:
~1,000 years,
~1GGb RAM.
(now down to seconds)

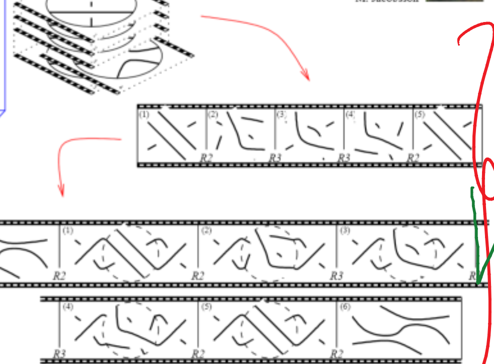
$\dim_j H_r$ is given by:

$j \backslash r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

the alternating knots zone

Functoriality / cobordisms.

M. Jacobsson 

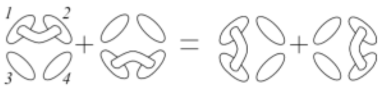


clean

J. Rasmussen: Leads to a no-analysis proof of a conjecture by Milnor.

A more general theory:

Remove G and NC, add

4Tu: 

(minor further revisions are necessary)

"God created the knots,
all else in topology is the work of mortals"
Leopold Kronecker (modified)

Visit!
Edit!
http://katlas.org